MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1933 APRIL 12

Prof. F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Samuel Alfred Barrett, F.R.G.S., B.S., M.S., Ph.D., Sc.D., Director of Milwaukee Public Museum, Wisconsin, U.S.A. ;

Vernon Brown, Wing Commander, R.A.F., M.A., A.F.R.Aë.S., Eversley, Sollershott West, Letchworth, Herts ;

George Douglas Buchanan, Nether Kirkton, Neilston, Renfrewshire, Scotland ;

Herbert Frank Finch, Royal Observatory, Greenwich, London, S.E. 10 ;

Walter James Langford, M.Sc., B.Sc., Senior Mathematical Master, Bec School, Tooting, 12 Stratton Road, Merton Park, London, S.W. 19 ;

Henri Mineur, D.-ès-Sc., Astronome à l'Observatoire de Paris, 16 Avenue Trudaine, Paris 9, France ; and

Andrew David Thackeray, B.A., King's College, Cambridge.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

William Rhys Evans, B.Sc., A.M.I.E.E., Parkfield, Cannock Road, Stafford (proposed by Peter Doig).

Ninety-six presents were announced as having been received since the last meeting, including :—

Dearborn Observatory, Annals, vol. iv, pt. 4.
Princeton Observatory, A Multiplet Table of Astrophysical Interest, by Charlotte E. Moore.
Rev. C. Whyte, Stellar Wonders.

THE MEAN FREE PATH IN RARE IONIZED GASES.

V. C. A. Ferraro, Ph.D.

1. A knowledge of the mean free path in a rare ionized gas is of interest in connection with several astrophysical and geophysical problems: its length has often been calculated by treating the particles in the gas as neutral, and though, as we shall see, this is true if the temperature is high or if the proportion of neutral to ionized atoms is comparatively large, the length of the free path has to be reduced considerably in the case of a rarefied ionized gas, such as the Sun's atmosphere, on account of the charge on the particles.

It may seem at first sight impossible to assign a meaning to the mean free path in the case of a gas consisting of particles which behave like point-centres of force instead of as rigid elastic spheres. Nevertheless when only orders of magnitude are concerned there are two methods of defining the mean free path to be associated with transport phenomena. The first consists * in comparing the formulæ for the coefficient of diffusion for the case of rigid elastic spheres and for the case of particles attracting according to the inverse square law; the second method consists in determining the time taken for a particle to be deflected through an angle δ, say, which is usually taken to be round about 90°. Such a deflection may be produced by a single encounter of two particles or by a succession of feebler binary encounters in which individually the deflection of one particle is less than δ but which produce a resultant deflection equal to δ. In the case of rare gases such successions of feeble encounters are more effective in deflecting the particle concerned than are single encounters. This was first pointed out by Sir James Jeans † in connection with collisions of stars with one another. It will be here shown that both methods lead to formulæ for the mean free path which are approximately the same.

We begin by considering the second method.

2. The ionized gas will be supposed at a uniform temperature and pressure throughout and, in the first instance, completely ionized; later (§ 6) we shall consider the effect of the presence of neutral atoms. The forces between the particles will be mainly electrostatic and will vary according to the inverse square law. It will also be supposed that the gas is electrically neutral. Denoting the number-density, mass and charge of a positive ion by n_i, m_i, Ze, and the corresponding quantities for an electron by n_e, m_e, $-e$ (e being measured in e.s.u.), we must have

$$n_e = Zn_i. \qquad (1)$$

* *M.N.*, 89, 79, 1928. † *Problems of Cosmogony and Stellar Dynamics*, p. 224, 1919.

In the first instance we shall neglect the encounters between the electrons themselves and consider only the encounters between the electrons and ions. Let g denote the relative velocity of an electron and an ion before encounter, b the perpendicular distance between the asymptotes,* δ the deflection of the relative path after the encounter: then we have

$$b = (Ze^2 \cot(\delta/2))\left(\frac{1}{m_i}+\frac{1}{m_e}\right)\Big/ g^2; \tag{2}$$

we must now adopt some mean value for g^2. If we neglect m_e/m_i in comparison with unity, then g will be approximately the velocity of the electron before encounter, and it seems sufficient for our purpose to take the mean value of g^2 to be the mean square of the individual velocities, viz.

$$g^2 = 3kT/m_e; \tag{3}$$

the same assumption is made by Professor Chapman in his calculation of the coefficient of diffusion †; (2) then becomes

$$b = Ze^2 \cot(\delta/2)/3kT. \tag{4}$$

Thus the probability that during an encounter between an electron and a positive ion the path of the electron is turned through an angle greater than or equal to δ after encounter is such that the electron must be aimed at the ion within a perpendicular distance equal to b. Since, on the whole, the motion of the ions can be ignored during encounters, the number of such encounters per electron per unit time is given by

$$\pi n_i b^2 \bar{c}_e, \tag{5}$$

$\bar{c}_e$ being the mean velocity of the electrons: the average duration τ of a free path is thus given by

$$\tau = 1/\pi n_i b^2 \bar{c}_e. \tag{6}$$

We must now correct this expression by considering the cumulative effect produced by a series of feeble encounters: the analysis was given by Sir James Jeans ‡ some years ago, and the results given here have been modified slightly to suit the present problem.

Neglecting the motion of the ion during an encounter with an electron the relative velocity g of the electron and ion before encounter will not differ appreciably from the velocity of the electron. As shown by Jeans (*loc. cit.*, pp. 225–227) the total change of velocity along the line of shortest approach is $2Ze^2/bgm_e$: the component of this velocity, c_n say, perpendicular to the direction of the relative velocity g is $2Ze^2 \cos(\delta/2)/bgm_e$. Since all directions of the plane containing g and the line of shortest approach are equally probable, the expectation of c_n^2 after a series of encounters is

$$c_n^2 = \sum\left(\frac{2Ze^2}{bgm_e}\right)^2 \cos^2(\delta/2). \tag{7}$$

* Only hyperbolic orbits need be considered. See § 3.

† *M.N.*, **82**, 292, 1922.

‡ *Problems of Cosmogony and Stellar Dynamics*, pp. 224–229, 1919.

Eliminating δ from this equation by using (4) we find

$$c_n^2 = 4\sum\left(\frac{Ze^2}{gm_e}\right)^2 \cdot \frac{1}{b^2+\mu^2}, \tag{8}$$

where

$$\mu = Ze^2/3kT. \tag{9}$$

It is sufficient to replace g^2 by its mean value given by (3) in the equation (8) since the effect of encounters for which g is small can be neglected. In an interval τ', the number of encounters for which b lies between b and $b+db$ is, from (5), equal to $2\pi n_i \bar{c}_e b db \tau'$, and hence from (8), if b varies from b_1 to b_2 during time τ', the expectation of c_n^2 after time τ' is

$$c_n^2 = \int_{b_1}^{b_2} 8\pi n_i \bar{c}_e \left(\frac{Ze^2}{gm_e}\right)^2 \tau' \cdot \frac{bdb}{b^2+\mu^2} = 4\pi n_i \bar{c}_e \left(\frac{Ze^2}{gm_e}\right)^2 \tau' \log_e \frac{b_2^2+\mu^2}{b_1^2+\mu^2}. \tag{10}$$

We may take the lower limit $b_1 = 0$; as regards the upper limit b_2 we adopt the procedure used by Professor Chapman * in his calculation of the coefficient of diffusion: in this b_2 is taken equal to the mean distance apart of the particles in the gas, viz. $(n_i+n_e)^{-1/3}$. It amounts to neglecting ternary and multiple encounters and this procedure appears to be justified.†

We now define a collision between an electron and an ion: this we shall suppose to occur when the direction of motion of the electron is turned through a right angle. For this to happen c_n^2 must be comparable with c_e^2 or, say, g^2; the free path of the electron, λ_e' say, is given by $\bar{c}_e\tau'$, and supposing for the moment that $\tau' < \tau$, we deduce from (10) the following expression:—

$$\lambda_e' = \frac{1}{4\pi n_i}\left(\frac{3kT}{Ze^2}\right)^2 \frac{1}{\log_e(1+a^2)}, \tag{11}$$

where

$$a = b_2/\mu = (3kT/Ze^2)(n_i+n_e)^{-1/3}. \tag{12}$$

3. In deriving this expression we have not taken into account the encounters between the electrons themselves; but in considering phenomena depending on the diffusion between ions and electrons, encounters between like particles are rightly neglected so that (11) gives the value of the mean free path for the case of diffusion. We now compare (11) with the expression for the mean free path deduced from the coefficient of diffusion D.

Regarding the particles as rigid elastic spheres we have

$$D = \frac{1}{3}\frac{n_e\lambda_i\bar{c}_i + n_i\lambda_e\bar{c}_e}{n_i+n_e}; \tag{13}$$

supposing for the present that the electrons contribute almost entirely to the diffusion, this expression is approximately equal to

$$D = \frac{1}{3}\cdot\frac{n_i}{n_i+n_e}\lambda_e\bar{c}_e. \tag{14}$$

* *M.N.*, **82**, 294, 1922. † E. Persico, *M.N.*, **86**, 93, 1926.

Another expression for D, obtained by Professor Chapman,* is approximately

$$D=\frac{3}{n_i+n_e}\left(\frac{\pi}{2}\frac{m_i+m_e}{m_i m_e}\right)^{\frac{1}{2}}\frac{(kT)^{5/2}}{2\pi Z^2e^4\log_e(1+a^2)}, \tag{15}$$

a being given by (12). Equating this to (14) and neglecting the ratio m_e/m_i in comparison with unity, we deduce the following expression for the mean free path of the electrons :—

$$\lambda_e=\frac{9}{n_i\bar{c}_e}\left(\frac{\pi}{2m_e}\right)^{\frac{1}{2}}\frac{(kT)^{5/2}}{2\pi Z^2e^4\log_e(1+a^2)}. \tag{16}$$

Since $\bar{c}_e=(8\overline{c_e^2}/3\pi)^{\frac{1}{2}}=2(2kT/\pi m_e)^{\frac{1}{2}}$, this becomes

$$\lambda_e=1/8n_i(Ze^2/3kT)^2\log_e(1+a^2), \tag{17}$$

which differs from (11) only by the numerical factor $\pi/2$. Thus both methods give approximately the same value for the mean free path of the electrons, and as far as orders of magnitude are concerned, we may take this free path to be given by (17).

Both (17) and (11) are deduced on the supposition that the orbits described by the electrons round a positive nucleus are hyperbolæ : this will only be the case if the mean kinetic energy of an electron $3kT/2$ is greater than its potential energy Ze^2/b_2 at a distance b_2 from the ion, where b_2 is the mean distance apart of the particles in the gas. Otherwise the electron is necessarily bound to an elliptic orbit. From (12) we see that this implies that $a>2$; thus the preceding analysis begins to break down when a becomes of order unity, since the ordinary methods of the kinetic theory of gases are then no longer applicable. The above formulæ may thus only be used when $a>>1$. Inserting numerical values for $k(=1{\cdot}37\,.\,10^{-16}$ c.g.s.) and $e(=4{\cdot}77\,.\,10^{-10}$ e.s.u.) the condition becomes

$$9{\cdot}10^2T/Z(n_i+n_e)^{\frac{1}{3}}>>1. \tag{18}$$

We shall now see that this condition makes $\tau'<\tau$, a condition assumed in deriving (11) : τ'/τ is the ratio of the time taken for an electron to be deflected through a right angle by a series of feeble encounters to the time taken for such a deflection to occur at a single encounter; we have $\tau'/\tau=\lambda_e'/\tau\bar{c}_e$, and from (4), (6), (11) (putting $\delta=\pi/2$) we find $\tau'/\tau=1/4\log_e(1+a^2)$. Since $a>>1$ we must have $\tau'<\tau$; for example, if $Z=1$, $T=6000°$, $n_i+n_e=10^6$, we find $\tau'/\tau=1/87$ approximately.

4. In dealing with a phenomenon such as viscosity in which collisions between like particles are important, the expressions (11) and (17) give a mean free path for the electrons which is too long. To correct this we must consider encounters between the electrons.

Let g_e denote the relative speeds of two encountering electrons before encounter ; then if, as seems plausible, we suppose that all possible directions

* *Loc. cit.*, p. 294.

of the velocities of the electrons before encounter are equally probable, taking the mean value of these velocities to be $(\overline{c_e^2})^{1/2}$, we find

$$g_e^2 = 2\overline{c_e^2} = 6kT/m_e, \tag{19}$$

and the path of one electron relative to the other * is turned through an angle greater than or equal to δ after collision if the paths of the electrons before encounter are at a perpendicular distance b_e apart given by

$$b_e = e^2 \cot(\delta/2)/3kT = b/Z. \tag{20}$$

This shows that an electron may suffer just as large a deflection through an encounter with another electron as with a positive ion. The number of collisions between the electrons per electron per unit time is then $\sqrt{2}\pi n_e b_e^2 \bar{c}_e$.

As far as orders of magnitude are concerned we may estimate the decrease in the length of the free path of the electrons by increasing the number of collisions per electron per unit time from the value given by (5) to $\pi n_i b^2 \bar{c}_e + \sqrt{2}\pi n_e b_e^2 \bar{c}_e$ or, by (1) and (20), to

$$\pi n_i b^2 \bar{c}_e (1 + \sqrt{2}/Z), \tag{21}$$

and applying the preceding analysis, which, however, is now only approximate. It amounts to decreasing (11) in the ratio $1 : 1 + \sqrt{2}/Z$; the correction is thus only of importance if Z is 1 or 2 but not greater.

We next deduce an expression for λ_i, the mean free path of the positive ions. Considering in the first instance only encounters between positive ions, the path of one ion relative to the other is turned through an angle greater than or equal to δ after collision if the paths of the ions before encounter are at a perpendicular distance b_i apart given by

$$b_i = Z^2 e^2 \cot(\delta/2)/3kT = Zb. \tag{22}$$

Proceeding as for the case of the electrons we find the following expression for λ_i, which is, however, only approximate:

$$\lambda_i = \frac{1}{4\sqrt{2}\pi n_e} \left(\frac{3kT}{Z^2 e^2}\right)^2 \frac{1}{\log_e (1 + a^2/Z^2)}; \tag{23}$$

if $Z = 1$, we have $\lambda'_e \sim \lambda_i$, *i.e.* the lengths of the mean free path of the ions and electrons are (approximately) equal.

We must now consider the effect of the encounters with the electrons: it is clear that this will shorten the free path since the number of collisions which the ions make per unit time is now increased by the presence of the electrons; but it seems unlikely that the reduction can be greater than by a factor of order unity. For the average magnitude of the impulse communicated by the electron to an ion is of order $m_e \bar{c}_e$ and hence proportional to $m_e^{\frac{1}{2}}$; the number of encounters per unit time per ion is proportional to $\bar{c}_e$ and hence to $m_e^{-\frac{1}{2}}$: thus in unit time the total mean impulse of the electrons encountering a positive ion is independent of m_e. Likewise the

* This is a hyperbola with the "fixed" electron at the focus not enclosed by the orbit.

total mean impulse of the ions encountering with one another in unit time is independent of m_i, and since $b_i = Zb$ it is seen that the two total impulses in unit time must be of the same order of magnitude at most. We accordingly take (23) as giving the right order of magnitude of the mean free path of the positive ions; this justifies the approximation made in deriving (14) from (13) because $\bar{c}_i << \bar{c}_e$.

5. Since (11), (17) and (23) are only valid if $a >> 1$ we may replace the factor $\log_e(1+a^2)$ by $2\log_e a$ in (11) and (17) and the factor $\log_e(1+a^2/Z^2)$ by $2\log_e a/Z$ in (23). Inserting numerical values for the constants e and k we find approximately

$$\lambda_e = \frac{2{\cdot}72 \,.\, 10^4 T^2}{n_i Z^2\{1+0{\cdot}31 \log (T/Z(n_i+n_e)^{\frac{1}{3}})} \tag{24}$$

in cm.; and by (23) the mean free path of the positive ions is approximately

$$\lambda_i = \frac{1{\cdot}92 \,.\, 10^4 \,.\, T^2}{n_e Z^4\{1+0{\cdot}31 \log (T/Z^2(n_i+n_e)^{\frac{1}{3}})} \text{ cm.} \tag{25}$$

The factor $\{1+0{\cdot}31 \log (T/Z(n_i+n_e)^{\frac{1}{3}}\}^{-1}$ varies very slightly over a large range of values of $T/Z(n_i+n_e)^{\frac{1}{3}}$ (provided $a >> 1$); for instance, the value of the factor for values of $T/Z(n_i+n_e)^{\frac{1}{3}}$ equal to $1/10$, 10^3, 10^{10} are respectively $1{\cdot}45$, $0{\cdot}52$, $0{\cdot}24$, or roughly in the ratio $6:2:1$. Hence to a first approximation $\lambda_e \propto T^2/Z^2 n_i$; it therefore increases rapidly with the temperature. It cannot, however, become greater than the value, $(\lambda_e)_0$ say, given by

$$(\lambda_e)_0 = 1/\sqrt{2}\pi n_e \sigma^2, \tag{26}$$

where σ is the diameter of the positive ion present in the gas. For instance, taking as a mean value $\lambda_e = 4{\cdot}10^4 T^2/Z^2 n_i$, the ratio $\lambda_e/(\lambda_e)_0 = (4\sqrt{2})\pi 10^4 \sigma^2 T^2/Z$; and supposing m_i is 40 times the mass of the hydrogen atom, $Z=1$, and $\sigma = 3{\cdot}10^{-8}$ cm., this ratio becomes of the order $10^{-10}T^2$. Hence the free path given by (24) becomes comparable with $(\lambda_e)_0$ if T is about $10^{5\circ}$.

It is of interest to compare the value of λ_e and $(\lambda_e)_0$ in an ionized atmosphere such as the solar calcium chromosphere: taking $T = 6000^\circ$, $n_i = n_e = 10^6$ ion/c.c., λ_e is found to be about 6·2 km. and the free path of the calcium ions is approximately the same. Calculating the mean free path by the formula (26) the mean free paths are found to be about 2500 km., or about 400 times as long as the value deduced from the formula (24).

6. If neutral atoms are present in the gas, the mean free paths of the ions and electrons will be reduced since the frequency of collisions is thereby increased. Both the ions and electrons will be deflected appreciably in colliding with a neutral atom, and we may estimate the reduction in the free paths by comparing the frequency of the encounters of the ions (or electrons) with the neutral atoms, with the frequency of the encounters of the ions (or electrons) with the ions. This amounts to comparing λ_i (or λ_e) with λ, the mean free path of the neutral particles: the latter may be taken as approximately given by

$$\lambda = 1/\sqrt{2}\pi\sigma^2 n, \tag{27}$$

where n is the number density of the neutral atoms. A correction should strictly be made for the electrostatic interaction of the charged particles with the neutral atoms, but this seems likely to underestimate only slightly the effect of the presence of neutral atoms.

We shall illustrate by considering the Sun's reversing layer. Neglecting the variation in temperature in the layer and taking as a mean value $T = 5500°$ we find from (24) that the free path of the electrons is approximately $3{\cdot}6 \cdot 10^{11}/n_e(1 - 0{\cdot}048 \log n_e)$ cm. ; at a depth in the reversing layer where the electron pressure is 30 dynes/cm.^{2} (that is, near the base), corresponding to $n_e = 4 \cdot 10^{13}$ electrons/c.c., this reduces to 0·025 cm. The mean free path of the neutral atoms as given by (27) is $2 \cdot 10^{14}/n$ cm. approximately (taking $\sigma = 3 \cdot 10^{-8}$ cm.), and becomes comparable with the mean free path of the electrons if n is of the order 10^{16} atoms/c.c., *i.e.* if the proportion of neutral to ionized atoms is about 250 : 1.

According to H. N. Russell,* the proportion of neutral (hydrogen) atoms to ionized atoms in the Sun's atmosphere is about 100 : 1, so that the free path obtained by ignoring the charges on the ions is of the same order of magnitude as that calculated by treating the particles as neutral, provided that this is calculated from the *total* pressure in the layer and not from the electron pressure.

THE PULSATION THEORY OF CEPHEID VARIABLES.

J. A. Edgar, M.A.

(Communicated by Sir Arthur Eddington)

1. Various criticisms of the pulsation theory of Cepheids have been advanced from time to time, and one of some importance was urged by Shinjo and Jeans, namely, that harmonics with one or more nodes in the interior of the star are possible as well as the fundamental vibration which is the only one observed. This paper, which was written in order to give a detailed examination of this criticism, falls into four sections : firstly, a brief statement of the fundamental equations of the theory is given ; secondly, two numerical solutions are found for both the fundamental and first harmonic vibrations ; thirdly, the dissipation of energy and time of decay of both vibrations are calculated, the term due to the opacity being treated separately to see the effect of lowering the index in Kramers's opacity coefficient ; and fourthly, the stability of the star is examined when pulsating in its fundamental and first harmonic vibrations.

2. *Fundamental Equations.*—In the derivation of the equation of the pulsation theory we shall follow Rosseland's method † of forming the variational equations from the non-linear equations for the conservation of mass, momentum and energy. Consider a fluid mass moving under the

* H. N. Russell, *Astrophysical Journal*, 75, 337, 1932.

† Rosseland, *Publ. Univ. Obs. Oslo*, No. 1, 1931.

influence of its own gravitation. If ρ, $\mathbf{V}$ denote density and the vectorial velocity of the fluid at any time, then the usual hydrodynamical equation for the conservation of mass is

$$\frac{d\rho}{dt}+\rho \text{ div } \mathbf{V}=0, \tag{2.1}$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+(\mathbf{V} \cdot \nabla)$ is the usual total differential operator.

If the stress tensor $\mathbf{P}$ has components, in rectangular co-ordinates,

$$P_{ik}=\delta_{ik}(p+\tfrac{2}{3}\mu \text{ div } \mathbf{V})-\mu\left(\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_i}\right),$$

where

$$\delta_{ik}=\begin{cases}1, & i=k,\\ 0, & i\neq k,\end{cases}$$

and

μ = coefficient of viscosity,
p = hydrostatic pressure of fluid,

then the equation of conservation of momentum gives *

$$\rho\frac{d\mathbf{V}}{dt}=\rho\nabla\phi-\text{div } \mathbf{P}, \tag{2.2}$$

where ϕ is the gravitational potential.

Let E denote the density of internal energy of the fluid, $\mathbf{F}$ the flux of energy due to the diffusive motion of the energy carrying entities, then the total flux of energy due to mass motion and diffusion will be

$$E\mathbf{V}+\mathbf{F}.$$

Then, if

Φ = heat energy generated by friction

and

$-p$ div $\mathbf{V}$ = mechanical energy transformed into heat,

the equation for conservation of energy gives

$$\frac{\partial E}{\partial t}+\text{div } (E\mathbf{V}+\mathbf{F})=\Phi-p \text{ div } \mathbf{V}, \tag{2.3}$$

where

$$\Phi=-\tfrac{2}{3}\mu(\text{div } \mathbf{V})^2+\mu\sum_{i,\,k}\left(\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_i}\right)\cdot\frac{\partial V_i}{\partial x_k}.$$

Since energy is being generated throughout the medium at a prescribed rate we separate out of E a term E_s such that

$$\frac{\partial E_s}{\partial t}+\text{div } (E_s\mathbf{V})=-\Sigma, \tag{2.4}$$

where Σ is the amount of energy per unit time and volume converted into thermal or mechanical energy.

* Jeffreys, *Cartesian Tensors*.

To form the variational equations of motion from these non-linear equations we consider the radial oscillations of the gaseous star, the displacement being small in the interior of the star. We write in a general way $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, etc., where the index zero refers to the undisturbed state of the medium. Write further

$$\eta = \int V dt$$

and consider small oscillations about the equilibrium state, p', ρ', etc., being small.

Then to the first order of small quantities (2.2) becomes, neglecting viscosity variation,

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} = -\rho' g_0 - \rho_0 g' - \frac{\partial p'}{\partial r}, \tag{2.5}$$

where r is the distance from the centre of the star.

Both g_0 and g' satisfy Poisson's equation

$$\begin{aligned} \text{div } g_0 &= 4\pi G \rho_0, \\ \text{div } g' &= 4\pi G \rho', \end{aligned} \tag{2.6}$$

where G is the constant of gravitation.

Since

$$\rho' = -\text{div } \rho_0 \eta \tag{2.7}$$

and g' and η are both zero at the centre, therefore

$$g' = -4\pi G \rho_0 \eta. \tag{2.8}$$

Taking radiation-pressure into account we have with the usual notation

$$p = R\rho T + \tfrac{1}{3} a T^4 \tag{2.9}$$

and the equation of energy is

$$\frac{\partial}{\partial t}\{E' + (E_0 + p_0) \text{ div } \eta + \eta \nabla E_0\} = \Sigma' - \text{div } \mathbf{F}', \tag{2.10}$$

where the flux of heat is primarily radiative.

Now assume energy and pressure to be arbitrary functions of temperature and density so that

$$E' = E_1 T' + E_2 \rho', \quad \text{where} \quad E_1 = \frac{\partial E}{\partial T}, \qquad E_2 = \frac{\partial E}{\partial \rho}, \text{ etc.}$$

and

$$p' = p_1 T' + p_2 \rho'.$$

From (2.7) eliminating ρ' we get

$$T' = p_1^{-1}(p' + p_2 \text{ div } \rho_0 \eta)$$

and

$$E' = (E_1 p_1^{-1}) p' + \text{div } \rho_0 \eta (E_1 p_1^{-1} p_2 - E_2).$$

Introducing this expression in (2.10) and writing

$$\nabla E_0 = \frac{\partial E_0}{\partial p_0}\nabla p_0 = -\rho_0 g_0 \frac{\partial E_0}{\partial p_0},$$

$$\nabla \rho_0 = \frac{\partial \rho_0}{\partial p_0}\nabla p_0 = -\rho_0 g_0 \frac{\partial \rho_0}{\partial p_0},$$

$$\gamma = p_1 E_1^{-1}\left(1 + \frac{E_0}{p_0}\right) + (p_2 - p_1 E_1^{-1} E_2)\frac{\rho_0}{p_0} \tag{2.11}$$

and

$$\lambda = p_1 E_1^{-1}\frac{\partial E_0}{\partial p_0} + (p_2 - p_1 E_2 E_1^{-1})\frac{\partial \rho_0}{\partial p_0} \tag{2.12}$$

we get

$$p' = -\gamma p_0 \operatorname{div} \eta + \lambda \rho_0 g_0 \eta, \tag{2.13}$$

and introducing this expression into (2.5) we obtain finally

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} = \frac{4 g_0 \rho_0}{r}\eta + \frac{\partial}{\partial r}\{(1-\lambda)\rho_0 g_0 \eta\} + \frac{\partial}{\partial r}\{\gamma p_0 \operatorname{div} \eta\}. \tag{2.14}$$

In the case of the standard model we have

$$E = \alpha_1 R\rho T + aT^4,$$
$$p = R\rho T + \tfrac{1}{3}aT^4,$$

where a is Stefan's constant and $1 + \alpha_1^{-1}$ is the ordinary ratio of specific heats. If we write as usual

$$\tfrac{1}{3}aT_0^4 = (1-\beta)p_0,$$
$$\Gamma = 1 + \alpha_1^{-1},$$

we easily obtain from the derivation of γ above

$$\frac{\gamma - \frac{4}{3}}{\Gamma - \frac{4}{3}} = \frac{4 - 3\beta}{1 + 12(\Gamma - 1)(1-\beta)/\beta}, \tag{2.15}$$

defining γ as a weighted mean.

Also $\lambda = 1$ always.

Putting $\frac{\eta}{r} = \xi_1$ and $r = \xi_0$ in (2.14) and allowing for a time variation of period $\frac{2\pi}{n}$, we get Eddington's equation

$$\frac{d^2\xi_1}{d\xi_0^2} + \frac{4-\mu}{\xi_0}\frac{d\xi_1}{d\xi_0} + \left\{\frac{n^2\xi_0}{\gamma p_0} - \left(3 - \frac{4}{\gamma}\right)\frac{\mu}{\xi_0^2}\right\}\xi_1 = 0, \tag{2.16}$$

where

$$\mu = \frac{g_0 \rho_0 \xi_0}{p_0} \qquad \text{(not viscosity "}\mu\text{")}$$

$$= -4\frac{z}{u}\frac{du}{dz} \qquad \textit{I.C.S.}, (127.6)\ *$$

and u and z are respectively the argument and variable in Emden's equation.

* Eddington, *Internal Constitution of the Stars*, chap. viii.

Also

$$\frac{\rho_0}{p_0} = \frac{1}{u}\left(\frac{\rho_0}{p_0}\right)_c,$$

where c denotes central values.

Let

$$\omega^2 = \frac{n^2}{\gamma}\left(\frac{\rho_0}{p_0}\right)_c,$$

$$\alpha = 3 - \frac{4}{\gamma}. \tag{2.17}$$

Then Eddington's equation may be written

$$\frac{d^2\xi_1}{d\xi_0^2} + \frac{4-\mu}{\xi_0}\frac{d\xi_1}{d\xi_0} + \left(\frac{\omega^2}{u} - \frac{\alpha\mu}{\xi_0^2}\right)\xi_1 = 0. \tag{2.18}$$

This has been derived on the main assumptions, (1) that the oscillations are nearly adiabatic and (2) that the change of state due to the oscillations

TABLE I

Fundamental Vibration

$\alpha = 0{\cdot}1$ $\omega^2 = 0{\cdot}315$

ξ_0	ξ_1	$d\xi_1/d\xi_0$	$d^2\xi_1/d\xi_0^2$	$-\rho_1$	$-d\rho_1/d\xi_0$
0·0	1·0000	0	·0206	3·0000	0
½	1·0028	·0104	·0212	3·0136	·0522
1	1·0104	·0211	·0225	3·0523	·1069
1¼	1·0163	·0268	·0237		
1½	1·0238	·0329	·0249	3·1208	·1690
1¾	1·0328	·0393	·0263		
2	1·0434	·0461	·0278	3·2224	·2400
2¼	1·0558	·0532	·0293		
2½	1·0701	·0607	·0309	3·3621	·3200
2¾	1·0862	·0686	·0324		
3	1·1044	·0769	·0337	3·5439	·4087
3¼	1·1247	·0855	·0348		
3½	1·1472	·0943	·0353	3·7716	·5004
3¾	1·1719	·1031	·0351		
4	1·1988	·1117	·0333	4·0432	·5800
4¼	1·2278	·1197	·0291		
4½	1·2587	·1260	·0201	4·3431	·5940
4¾	1·2908	·1289	·0017		
5	1·3231	·1251	−·0367	4·5948	·3169

This gives a node at ξ_0 slightly greater than 5.

The corresponding solution with two nodes within the star was found to be given by $\omega^2 = 0{\cdot}30$.

is sufficiently small to justify the use of the linearisation process leading to the variational equations.

3. *Solution of the Equation* (2.18).—The equation (2.18) is solved first by a solution in series which presently diverges and then by a quadrature, using the condition that $d\xi_1/d\xi_0$ from symmetry is zero at the centre and $\rho_1 = -3\xi_1 - \xi_0 d\xi_1/d\xi_0$ is zero at a node.* The arbitrary value unity is given to ξ_1 at the centre. Tables are given for the fundamental and first harmonic

TABLE II

First Harmonic Vibration

$\alpha = 0{\cdot}1$ $\omega^2 = 0{\cdot}30$

ξ_0	ξ_1	$d\xi_1/d\xi_0$	$d^2\xi_1/d\xi_0^2$	$-\rho_1$	$-d\rho_1/d\xi_0$
0·0	1·0000	0·0000	− ·0330	3·0000	0·0000
0·2	·9992	− ·0068	− ·0351		
0·4	·9973	− ·0142	− ·0423		
0·5	·9957	− ·0188	− ·0459	2·9753	− ·0981
0·6	·9930	− ·0237	− ·0490		
0·8	·9859	− ·0355	− ·0586		
1·0	·9755	− ·0517	− ·0805	2·8748	− ·2873
1·2	·9633	− ·0707	− ·1115		
1·4	·9467	− ·0966	− ·1474		
1·5	·9363	− ·1121	− ·1684	2·6408	− ·7010
1·6	·9241	− ·1304	− ·1923		
1·8	·8938	− ·1741	− ·2476		
2·0	·8534	− ·2300	− ·3116	2·1002	− 1·5432
2·2	·8007	− ·3001	− ·3970		
2·4	·7331	− ·3889	− ·4937		
2·5	·6910	− ·4409	− ·5480	·9707	− 3·1336
2·6	·6448	− ·4985	− ·6046		
2·8	·5322	− ·6319	− ·7301		
3·0	·4103	− ·7925	− ·8822	− 1·1466	− 5·8166
3·2	·2334	− ·9835	− 1·0260		
3·4	·0152	− 1·2027	− 1·1667		
3·5	− ·1098	− 1·3222	− 1·2279	− 4·9016	− 9·5864
3·6	− ·2497	− 1·4481	− 1·2738		
3·8	− ·5655	− 1·7120	− 1·3308		
4·0	− ·9146	− 1·9776	− 1·3097	− 10·6542	− 13·1492
4·2	− 1·3356	− 2·2234	− 1·0995		
4·4	− 1·7994	− 2·3956	− ·5452		
4·5	− 2·0393	− 2·4280	− ·0979	− 17·0439	− 10·1525
4·6	− 2·2837	− 2·4133	·4522		
4·8	− 2·7469	− 2·1634	2·2197		
5·0	− 3·1178	− 1·4432	5·2343	− 16·4694	20·3981

This gives nodes at $\xi_0 = 2{\cdot}8$ and $\xi_0 > 5$.
For the purposes of this calculation $\alpha = 0{\cdot}4$ was considered.

* Eddington, *Internal Constitution of the Stars*, chap. viii.

vibrations corresponding to $\alpha = 0{\cdot}1$ and $\alpha = 0{\cdot}4$ (α is necessarily less than 0·6 for which value $\gamma = \frac{4}{3}$). In each case the node which falls near the surface is chosen well within the adiabatic region in which our equations hold. Table I is taken directly from Eddington's book,* and all the calculations refer to the Emden polytropic index 3.

TABLE III

Fundamental Vibration

$\alpha = 0{\cdot}4$ $\omega^2 = 0{\cdot}110$

ξ_0	ξ_1	$d\xi_1/d\xi_0$	$d^2\xi_1/d\xi_0^2$	$-\rho_1$	$-d\rho_1/d\xi_0$
0·0	1·0000	0·0000	·0860	3·0000	0·0000
0·2	1·0017	·0170	·0849		
0·4	1·0069	·0343	·0871		
0·5	1·0110	·0433	·0880	3·0546	·2172
0·6	1·0155	·0525	·0889		
0·8	1·0279	·0720	·0904		
1·0	1·0441	·0909	·1010	3·2232	·4646
1·2	1·0643	·1129	·1094		
1·4	1·0891	·1357	·1187		
1·5	1·1036	·1479	·1248	3·5826	·7788
1·6	1·1187	·1605	·1309		
1·8	1·1535	·1881	·1437		
2·0	1·1941	·2183	·1583	4·0189	1·1898
2·2	1·2421	·2519	·1744		
2·4	1·2959	·2885	·1923		
2·5	1·3261	·3085	·2016	4·7495	1·7380
2·6	1·3575	·3289	·2111		
2·8	1·4279	·3731	·2309		
3·0	1·5073	·4213	·2518	5·7858	2·4406
3·2	1·5967	·4735	·2727		
3·4	1·6969	·5301	·2920		
3·5	1·7521	·5600	·3001	7·1163	3·2904
3·6	1·8089	·5903	·3077		
3·8	1·9331	·6531	·3178		
4·0	2·0700	·7159	·3155	9·0736	4·1256
4·2	2·2194	·7775	·2949		
4·4	2·3806	·8341	·2760		
4·5	2·4698	·8527	·2112	11·2461	4·3612
4·6	2·5614	·8813	·1560		
4·8	2·7607	·8909	−·0610		
5·0	2·9420	·8576	−·2700	13·1240	2·0804

This gives a node at $\xi_0 > 5$.

* Eddington, *Internal Constitution of the Stars*, chap. viii.

TABLE IV

First Harmonic Vibration

$\alpha = 0{\cdot}4$ $\omega^2 = 0{\cdot}466$

ξ_0	ξ_1	$d\xi_1/d\xi_0$	$d^2\xi_1/d\xi_0^2$	$-\rho_1$	$-d\rho_1/d\xi_0$
0·0	1·0000	0·0000	·0148	3·0000	0·0000
0·2	1·0003	·0026	·0100		
0·4	1·0010	·0043	·0032		
0·5	1·0014	·0047	− ·0034	3·0065	·0171
0·6	1·0019	·0046	− ·0122		
0·8	1·0026	·0029	− ·0360		
1·0	1·0029	− ·0012	− ·0663	3·0075	− ·0711
1·2	1·0013	− ·0174	− ·0960		
1·4	·9959	− ·0409	− ·1399		
1·5	·9910	− ·0662	− ·1672	2·8737	− ·5156
1·6	·9846	− ·0745	− ·1991		
1·8	·9653	− ·1217	− ·2776		
2·0	·9348	− ·1866	− ·3737	2·4312	− 1·4938
2·2	·8893	− ·2726	− ·4901		
2·4	·8241	− ·3840	− ·6271		
2·5	·7824	− ·4504	− ·7033	+ ·5892	− 3·5596
2·6	·7337	− ·5246	− ·7808		
2·8	·6121	− ·6970	− ·9458		
3·0	·4527	− ·9027	− 1·1098	− 1·3500	− 6·9402
3·2	·2490	− 1·1392	− 1·2500		
3·4	− ·0043	− 1·3949	− 1·2729		
3·5	− ·1848	− 1·5184	− 1·2513	− 5·8628	− 10·4528
3·6	− ·3572	− 1·6420	− 1·2030		
3·8	− ·7081	− 1·8636	− ·9610		
4·0	− 1·0973	− 2·0075	− ·4120	− 11·3219	− 9·6780
4·2	− 1·5022	− 2·0147	·5609		
4·4	− 1·8887	− 1·7677	2·1716		
4·5	− 2·0525	− 1·4984	3·5313	− 12·9003	10·3952
4·6	− 2·1842	− 1·1036	4·6222		
4·8	− 2·2914	·1193	7·5828		
5·0	− 2·0909	2·0167	12·1339	4·5308	68·7363

This gives nodes at $\xi_0 = 2{\cdot}8$ and $\xi_0 = 4{\cdot}9$, and in this case the vibration does not appear to be symmetrical.

In the case of the fundamental vibration the displacement is fairly constant throughout the main interior of the star, while in the case of the first harmonic, whilst this property holds from the centre to the first node, there is a very great increase between the first and second nodes. This would lead us to expect that the first harmonic would be much more strongly dissipated towards the star's surface than the fundamental mode of vibration. In view of the subsequent working it is important to notice that relatively to the arbitrary central values the density and its gradient show much greater variations for the harmonic than for the fundamental. In both the fundamental and harmonic the variations are more pronounced for the larger α which corresponds to smaller γ and, therefore, from (2.15) to smaller Γ.

4. *Period of First Harmonic.*—In the above calculations an arbitrary value of the displacement is given at the centre, so that we have no idea of the relative values of the displacement for the fundamental and the first harmonic modes, which means that at present we have no idea whether the first harmonic is experimentally observable on the pulsation hypothesis. However the period, whether observable or not, can easily be given.

Let

π_1 = period of fundamental.
π_2 = period of first harmonic.
n_1 = frequency of fundamental $= \frac{2\pi}{\pi_1}$.
n_2 = frequency of first harmonic $= \frac{2\pi}{\pi_2}$.
G = constant of gravitation.

Then as in Eddington's *I.C.S.** we have

$$\pi_1^2\rho_c = \frac{10}{3}\frac{4\pi}{G\gamma\alpha}, \quad \text{where} \quad \rho_c = \text{central density},$$

$$\pi_2^2\rho_c = \frac{1}{3}\frac{4\pi}{G\gamma\alpha} \quad \text{for} \quad \alpha = 0{\cdot}1, \qquad \gamma = 1{\cdot}380$$

$$= \frac{1}{1{\cdot}165}\frac{4\pi}{G\gamma\alpha} \quad \text{for} \quad \alpha = 0{\cdot}4, \qquad \gamma = 1{\cdot}54.$$

$$\therefore \frac{\pi_2}{\pi_1} = \frac{n_1}{n_2} = \begin{cases} {\cdot}32 & \text{for} \quad \alpha = 0{\cdot}1 \\ {\cdot}51 & \text{for} \quad \alpha = 0{\cdot}4 \end{cases}. \tag{4.1}$$

For δ Cephei the theoretical value of π_1 is 3·57 days. Thus the theoretical of π_2 is somewhere near 1·4 days.

It is instructive to compare the above frequencies with those of a vibrating string which gives

$$\frac{n_1}{n_2} = 0{\cdot}5.$$

5. *Dissipation of Energy.*—For a discussion of the dissipation of energy it seems advisable to separate out that part of the dissipation which arises from a variation in the opacity of the material from the remainder. For this reason the discussion of *I.C.S.*, § 132,* is reproduced to some extent.

Let F = outward flow of radiation across a sphere of radius ξ.

$= F_0 + F_0F_1$, where F_0 = equilibrium value of F,
F_1 = variation in F due to pulsation.

$$\therefore F = -4\pi\xi^2\frac{ac}{3k\rho}\frac{dT^4}{d\xi}$$

$$= -\frac{4\pi ac\xi^4}{3k\rho_0{\xi_0}^2}\frac{dT^4}{d\xi_0}$$

by continuity of mass.

* Eddington, *Internal Constitution of the Stars*, chap. viii.

Hence, differentiating logarithmically,

$$F_1 = -k_1 + 4\xi_1 + \frac{4dT_0{}^4 T_1}{dT_0{}^4}. \tag{5.1}$$

But from § 2

$$P \propto \rho\gamma, \quad \text{where } P \text{ is the total pressure,}$$

$\therefore\ P_1 = \gamma\rho_1.$

Since

$$P = R\rho T + \tfrac{1}{3}aT^4$$

and

$$\tfrac{1}{3}aT_0{}^4 = (1-\beta)P_0,$$
$$R\rho T_0 = \beta P_0,$$

then

$$T_1 = \frac{\gamma-\beta}{4-3\beta}\rho_1, \qquad \text{by differentiating } P \text{ logarithmically,}$$

$$= (\gamma'-1)\rho_1, \tag{5.2}$$

where

γ = effective ratio of specific heats for the pressure-density relationship, and

γ' = effective ratio of specific heats for the temperature-density relationship, so that

$$\gamma' = \frac{4}{3} + \frac{\gamma - 4/3}{4-3\beta}.$$

From (5.1)

$$F_1 = -k_1 + 4\xi_1\left\{1 - \frac{1}{\eta}\left(4 + \frac{n^2\xi_0}{g_0}\right)\right\}, \tag{5.3}$$

where

$$\eta = \gamma/\gamma' - 1. \tag{5.4}$$

$\therefore$ with the absorption law ($k \propto \rho/T^{7/2}$)

$$k_1 = \rho_1 - \tfrac{7}{2}T_1 = -\theta\rho_1, \tag{5.5}$$

where

$$\theta = \tfrac{7}{2}(\gamma'-1) - 1. \tag{5.6}$$

Now let $\frac{dQ}{dt}$ be the rate of gain of heat per unit mass in the shell between ξ and $\xi + d\xi$ owing to the transfer of heat by radiation. Then

$$\frac{dQ}{dt} = -\frac{d(F_0 + F_0F_1)}{4\pi\rho_0\xi_0{}^2 d\xi_0}$$

$$= -\frac{1}{4\pi\rho_0\xi_0{}^2}(1+F_1)\frac{dF_0}{d\xi_0} - \frac{F_0}{4\pi\rho_0\xi_0{}^2}\frac{dF_1}{d\xi_0}.$$

The steady part on the right-hand side must be balanced by the rate of liberation of subatomic energy Σ in the shell. Hence

$$\frac{dQ}{dt} = -\Sigma(1+F_1) - \tfrac{1}{3}\Sigma_m\frac{\rho_m}{\rho_0}\xi_0\frac{dF_1}{d\xi_0}, \tag{5.7}$$

where Σ_m and ρ_m are mean values interior to ξ so that

$$\Sigma_m = F_0 / \tfrac{4}{3}\pi\rho_m\xi_0^3.$$

With a view to actual computation we must find $n^2\xi_0/g_0$ in (5.3) in terms of tabulated functions of Emden's equation.

If R is the star's arbitrary radius, then

$$\frac{n^2R}{g_0} = \frac{n^2R^2}{GM} = \frac{4\pi^2}{\pi_1^2} \cdot \frac{3}{4\pi G\rho_m} = \frac{3\pi}{\pi_1^2 G\rho_c}\left(\frac{\rho_c}{\rho_m}\right)_B$$

for the fundamental, where $\left(\frac{\rho_c}{\rho_m}\right)_B$ means the boundary value of $\frac{\rho_c}{\rho_m}$.

Hence from § 4

$$\frac{n^2R}{g_0} = 12{\cdot}2(\gamma\alpha) = 1{\cdot}71 \quad \text{for} \quad \alpha = 0{\cdot}1.$$

Since $\frac{n^2\xi_0}{g_0}$ is 1·71 at the boundary and at other points it is inversely proportional to the mean density interior to ξ_0, we find

$$\frac{n^2\xi_0}{g_0} = \frac{1{\cdot}71}{54{\cdot}18}\left(\frac{\rho_c}{\rho_m}\right)_0 \quad \text{for} \quad \alpha = 0{\cdot}1$$

$$= 0{\cdot}032D_0, \quad \text{where} \quad D_0 = \left(\frac{\rho_c}{\rho_m}\right)_0.$$

Similarly

$$\frac{n^2\xi_0}{g_0} = 0{\cdot}139D_0 \quad \text{for} \quad \alpha = 0{\cdot}4 \text{ for the fundamental.}$$

Proceeding in a similar way for the first harmonic, we may group the results as follows:—

$$\frac{n^2\xi_0}{g_0} = bD_0, \tag{5.8}$$

where

$$\begin{aligned} b &= 0{\cdot}031 \quad \text{for } \alpha = 0{\cdot}1, \quad \omega^2 = 0{\cdot}0315, \\ &= 0{\cdot}310 \quad \text{,, } \alpha = 0{\cdot}1, \quad \omega^2 = 0{\cdot}30, \\ &= 0{\cdot}139 \quad \text{,, } \alpha = 0{\cdot}4, \quad \omega^2 = 0{\cdot}110, \\ &= 0{\cdot}541 \quad \text{,, } \alpha = 0{\cdot}4, \quad \omega^2 = 0{\cdot}466. \end{aligned}$$

From (5.7) and (5.3), putting $\Sigma_m = \Sigma$,

$$\frac{dQ}{dt} = A + B - \Sigma, \tag{5.9}$$

where

$$A = -\Sigma\left(\theta\rho_1 + \frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\theta\frac{d\rho_1}{d\xi_0}\right) \tag{5.10}$$

and

$$B=\Sigma\left[\left\{\left(\frac{16}{\eta}-4\right)+\frac{4b}{\eta}D_0\right\}\left\{\xi_1+\frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\xi_1}{d\xi_0}\right\}+\frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{4b}{\eta}\xi_1\frac{dD_0}{d\xi_0}\right]. \quad (5.11)$$

It is important to notice that in the form of $\frac{dQ}{dt}$ in (5.9), since the law governing the rate of liberation of sub-atomic energy is unknown, we have assumed the rate is unchanged throughout the stellar interior.

From (5.5) and the subsequent working we easily see that A represents the rate of gain of heat per unit mass in the shell between ξ and $\xi+d\xi$ owing to the variation in the opacity of the material, while B represents the contribution directly due to the expansion and compression of the star during the process of pulsation.

With the aid of the *B.A. Mathematical Tables*, vol. ii, we obtain the following tables (in each case the values of the constants from §§ 2 and 5, together with A and B, are given for reference):—

TABLE V

Fundamental Vibration

$\alpha=0{\cdot}1$ $\omega^2=0{\cdot}0315$

ξ_0	A/Σ	B/Σ	Sum
0·0	·720	·134	·854
0·5	·726	·139	·865
1·0	·743	·156	·899
1·5	·780	·196	·976
2·0	·850	·276	1·126
2·5	·985	·489	1·474
3·0	1·240	·957	2·197
3·5	1·753	2·052	3·805
4·0	2·794	4·971	7·765
4·5	4·774	12·117	16·891
5·0	5·581	36·814	42·395

where

$$\gamma=1{\cdot}38,\quad (1-\beta)=\cdot385,\quad \eta=3{\cdot}90,\quad \theta=0{\cdot}24,\quad \gamma'=1{\cdot}355,$$

and

$$\Gamma=1{\cdot}43 \quad \text{(for fundamental and first harmonic).}$$

$$\therefore\ A=-\Sigma\left(\cdot24\rho_1+\cdot08\xi_0\frac{\rho_m}{\rho_0}\frac{d\rho_1}{d\xi_0}\right),$$

$$B=\Sigma\left[\left(\cdot102+\cdot032D_0\right)\left(\xi_1+\frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\xi_1}{d\xi_0}\right)+\frac{0{\cdot}032}{3}\frac{\rho_m}{\rho_0}\xi_0\xi_1\frac{dD_0}{d\xi_0}\right].$$

TABLE VI

First Harmonic Vibration

$\alpha = 0{\cdot}1$ $\omega^2 = 0{\cdot}30$

ξ_0	A/Σ	B/Σ	Sum
0·0	·720	·422	1·142
0·5	·710	·460	1·170
1·0	·674	·587	1·261
1·5	·508	·843	1·351
2·0	·015	1·185	1·200
2·5	− 1·483	1·022	− ·461
3·0	− 5·818	− 2·914	− 8·732
3·5	− 17·450	− 26·033	− 43·483
4·0	− 43·920	− 126·607	− 170·527
4·5	− 67·810	− 497·320	− 565·130
5·0	284·250	− 1307·550	− 1023·300

where

$$A = -\Sigma\left(\cdot 24\rho_1 + \cdot 08\xi_0 \frac{\rho_m}{\rho_0}\frac{d\rho_1}{d\xi_0}\right),$$

$$B = \Sigma\left[\left(\cdot 102 + \cdot 320 D_0\right)\left(\xi_1 + \frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\xi_1}{d\xi_0}\right) + \frac{0{\cdot}320}{3}\frac{\rho_m}{\rho_0}\xi_0\xi_1\frac{dD_0}{d\xi_0}\right].$$

For $\alpha = 0{\cdot}4$ we obtain the following tables :—

TABLE VII

Fundamental Vibration

$\alpha = 0{\cdot}4$ $\omega^2 = 0{\cdot}110$

ξ_0	A/Σ	B/Σ	Sum
0·0	2·040	1·160	3·200
0·5	2·075	1·278	3·353
1·0	2·320	1·353	3·673
1·5	2·834	1·732	4·566
2·0	3·802	2·608	6·410
2·5	5·929	4·752	10·681
3·0	10·525	10·209	20·734
3·5	20·640	25·033	45·673
4·0	42·940	67·440	110·380
4·5	85·269	197·840	283·109
5·0	92·015	599·020	691·035

where

$\gamma = 1{\cdot}54$, $(1-\beta) = 0{\cdot}14$, $\eta = 3{\cdot}21$, $\theta = 0{\cdot}68$, $\gamma' = 1{\cdot}48$, and $\Gamma = 1{\cdot}67$.

$$\therefore\ A = -\Sigma\left(\cdot 68\rho_1 + \frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0 \cdot 68\frac{d\rho_1}{d\xi_0}\right),$$

$$B = \Sigma\left[\left(\cdot 99 + \cdot 17D_0\right)\left(\xi_1 + \frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\xi_1}{d\xi_0}\right) + \frac{0{\cdot}17}{3}\frac{\rho_m}{\rho_0}\xi_0\xi_1\frac{dD_0}{d\xi_0}\right].$$

TABLE VIII

First Harmonic Vibration

$\alpha = 0{\cdot}4$ $\omega^2 = 0{\cdot}466$

ξ_0	A/Σ	B/Σ	Sum
0·0	2·040	1·660	3·700
0·5	2·047	1·787	3·834
1·0	2·026	2·073	4·099
1·5	1·686	2·462	4·148
2·0	·311	2·647	2·958
2·5	− 5·121	·510	− 4·611
3·0	− 19·660	− 13·135	− 32·795
3·5	− 54·150	− 69·391	− 123·541
4·0	− 93·960	− 249·772	− 343·732
4·5	176·330	− 544·760	− 368·430
5·0	2757·080	+ 743·300	3500·380

where

$$A = -\Sigma\left(\cdot 68\rho_1 + \frac{0{\cdot}68}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\rho_1}{d\xi_0}\right),$$

$$B = \Sigma\left[\left(\cdot 99 + \cdot 67D_0\right)\left(\xi_1 + \frac{1}{3}\frac{\rho_m}{\rho_0}\xi_0\frac{d\xi_1}{d\xi_0}\right) + \frac{0{\cdot}67}{3}\frac{\rho_m}{\rho_0}\xi_0\xi_1\frac{dD_0}{d\xi_0}\right).$$

From an examination of these tables we see that in the case of the fundamental mode the "A" and "B" terms are comparable in the main interior, while towards the node the "B" term becomes by far the more important. In the case of the first harmonic modes the terms are comparable until near the second node when the B term again becomes the greater, although Table VIII seems to indicate that the opacity term becomes very large and more important than the other after we pass through the second node. The general equality of the two terms has important applications in the next part of the paper. In view of the arbitrary value of the vibration at the centre of the star it is difficult to compare the cases $\alpha = 0{\cdot}1$ and $\alpha = 0{\cdot}4$; but, roughly, the tables seem to indicate that $\frac{dQ}{dt}$, and hence, as we shall see in a moment, the dissipation is greater for the larger value of α for both the harmonic and the fundamental modes, although the harmonic seems to suffer greater dissipation for $\alpha = 0{\cdot}1$ than $\alpha = 0{\cdot}4$ when we compare it with the fundamental. These conclusions will now be examined in detail.

6. The small leakage of heat found in the last section will gradually dissipate the energy of the pulsation if there is no counterbalancing energy to preserve the vibration. We shall estimate roughly the time of decay in all four cases.

Let pulsation be kept steady by supplying mechanical work W, so that by the conservation of energy

$$W+\int dQ=0 \tag{6.1}$$

for any number of complete cycles.

Now the change of entropy in a complete cycle is zero,

$$\therefore \int\frac{dQ}{T}=0 \quad \text{for such a cycle.}$$

$$\therefore \int dQ\left(\frac{1}{T_0}-\frac{T_1}{T_0}\right)=0 \quad \text{for such a cycle.}$$

$$\therefore W_1+\int TdQ=0, \tag{6.2}$$

which gives W correct to the second order of small quantities.

In terms of the central displacement $[\xi_1]_c$ we have, from (5.9),

$$\frac{dQ}{dt}=(A+B)[\xi_1]_c\cos nt,$$

and from (5.2)

$$T_1=(\gamma'-1)\rho_1[\xi_1]_c\cos nt.$$

Hence W (per gram per sec.)

$$=-\tfrac{1}{2}(\gamma'-1)\rho_1(A+B)[\xi_1]_c{}^2\Sigma$$
$$=W_1+W_2,$$

where

$$W_1=-\tfrac{1}{2}(\gamma'-1)\rho_1A[\xi_1]_c{}^2\Sigma$$

represents the dissipation due to the opacity, and

$$W_2=-\tfrac{1}{2}(\gamma'-1)\rho_1B[\xi_1]_c{}^2\Sigma$$

represents the remaining dissipation.

For $\alpha=0{\cdot}1$ in the case of the fundamental, we find by rough quadratures that the mean values of these are

$$\bar{W}_1=\cdot68[\xi_1]_c{}^2\Sigma,$$
$$\bar{W}_2=\cdot70[\xi_1]_c{}^2\Sigma,$$

and hence for the whole star

$$\bar{W}_1=\cdot68[\xi_1]_c{}^2L, \tag{6.3}$$
$$\bar{W}_2=\cdot70[\xi_1]_c{}^2L, \tag{6.4}$$

and

$$\therefore \bar{W}=1{\cdot}38[\xi_1]_c{}^2L, \tag{6.5}$$

where $L=$energy emitted by the star per second.

The kinetic energy of the pulsation per gram is given by

$$K.E.=\tfrac{1}{2}(n\xi_0\xi_1)^2.$$

By a rough quadrature the mean value of this for the fundamental ($\alpha = 0{\cdot}1$) is

$$\overline{K.E.} = {\cdot}067 n_1^2 R^2 [\xi_1]_c^2,$$

so that for the whole star (mass M),

$$\overline{K.E.} = {\cdot}067 n_1^2 R^2 [\xi_1]_c^2 M. \qquad (6.6)$$

By dividing (6.6) by (6.5) we obtain the time of decay T for the fundamental pulsation in this case.

Proceeding in the same way for the other cases and using the experimental results for δ Cephei * we obtain the following table:—

TABLE IX

α	ω^2	$\bar{W}_1/L[\xi_1]_c^2$	$\bar{W}_2/L[\xi_1]_c^2$	$\overline{K.E.}/R^2M[\xi_1]_c^2$	$\bar{W}_1/\bar{W}_2$	T
0·1	·0315	·68	·70	$\cdot 067 n_1^2$	·97	8000 years
	·30	7·74	58·00	$\cdot 090 n_2^2$	·13	2500 ,,
0·4	·110	15·77	29·11	$\cdot 14 n_1^2$	·54	8300 ,,
	·466	8·00	71·28	$\cdot 05 n_2^2$	·11	6600 ,,

where n_1 and n_2 are found from § 4.

Thus we see that in the main our previous inferences were correct, viz. that the harmonics are dissipated more than the fundamental vibration in a free pulsation of the system, the effect being more pronounced for the smaller value of α. The fundamental appears, however, to decay in about the same time for all values of α. Having regard to the supposed "life" of the stars, the above values of the time of decay appear to exclude the hypothesis of Cepheid variation being due to a free oscillation of the star, as long as the internal conditions in Cepheids obey the usual laws of opacity, etc. An examination of Table IX shows a method of altering these laws to make the duration of such a free oscillation longer, namely, by changing the sign in the third column. This would be brought about, remembering the definition of $\bar{W}_1$, by a reduction in the index x of the opacity relation

$$k \propto \rho / T^x,$$

x having previously been taken as $7/2$.

If x be taken as $2{\cdot}2$ we find that $\bar{W}_1$ is almost reversed in sign and the dissipation is practically zero for the fundamental in the case of $\alpha = 0{\cdot}1$. A reduction in x of this amount would not be sufficient to reverse the sign of $\bar{W}_1$ for $\alpha = 0{\cdot}4$. Thus we might have a free oscillation provided we take $x = 2{\cdot}2$ and take a small value of α and, therefore, of Γ the so-called ratio of the specific heats of the material of the star without radiation being taken into account. It is important to notice that this would also account for the

* Eddington, *Internal Constitution of the Stars*, chap. viii.

non-appearance of harmonics in the observed vibrations, as the reduction in x would decrease $\bar{W}_1$, which is, however, only a small fraction of $\bar{W}$ in this instance, so that T would still be comparatively small.

Since $k_1 = -\{x(\gamma' - 1) - 1\}\rho_1$ from the values of γ' given in previous paragraphs, we see that for $x = 7/2$ the opacity decreases with compression, while with $x < \frac{5}{2}$ and a low value of α it increases with the compression. The effect of this is that we may regard the star as a thermodynamic machine which works by varying the leak of heat. Some consequences of this have been considered.* The reduction in x, which we have indicated above, might be caused by a large value of $(1 - \beta)$, corresponding to small γ', instead of a low value of Γ. This would correspond to stars of large mass, a point in favour of this theory. Before leaving this subject we anticipate that on the whole a reduction in the index x would result in a retardation of the leakage wave in the same direction as the retardation of the non-adiabatic transition layers of the star.

7. *Stability of Star.*†—The above considerations of the times of decay of the fundamental and first harmonic modes of vibration of the star show that the star is stable for free vibrations of such a kind, but it is instructive to consider the question of stability directly from the equations of motion together with their solutions. Denoting perturbed functions by dashes and adopting a time factor $e^{i\sigma_k t}$ for the kth mode of vibration, the equation for the conservation of energy (2.3) gives

$$p'_k + \gamma p_0 \operatorname{div} \xi_k - \rho_0 g_0 \xi_k = -\frac{i(\gamma - 1)}{\sigma_k}\{\Sigma'_k - \operatorname{div} F'_k\}, \tag{7.1}$$

corresponding to the displacement of the kth mode ξ_k. For simplification we neglect light pressure and viscosity. However, the equation will only require a minor adjustment to apply to stars built according to the standard model.

The equation of motion is, similarly, from (2.2),

$$\frac{\partial p_k'}{\partial \xi_0} = g_0 \operatorname{div} \rho_0 \xi_k + 4\pi G \rho_0^2 \xi_k + \rho_0 \sigma_k^2 \xi_k. \tag{7.2}$$

Eliminating p_k' from (7.1) by means of (7.2) we obtain the slightly perturbed "self-adjoined" differential equation

$$\frac{\partial}{\partial \xi_0}(\gamma p_0 \operatorname{div} \xi_k) + (\sigma_k^2 + 4g_0/\xi_0)\rho_0 \xi_k = -\frac{i(\gamma - 1)}{\sigma_k}\frac{\partial}{\partial \xi_0}\{\Sigma_k' - \operatorname{div} F_k'\}. \tag{7.3}$$

Expanding the perturbing term on the right-hand side and the perturbation in amplitude in a series of the unperturbed amplitudes ξ_k^0, we obtain by the usual methods for the solution of such equations

$$i\sigma_k' = \frac{\gamma - 1}{2\sigma_k^{2_0}} \int_0^R \frac{\partial}{\partial \xi_0}\{\Sigma_k' - \operatorname{div} F_k'\}\xi^0{}_k \xi_0^2 d\xi_0.$$

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* Eddington, *Internal Constitution of the Stars*, chap. viii.

† Rosseland, Univ. Obs., Oslo, *Publication* No. 2, 1932.

ξ_1^0 is given in columns 2 of Tables I and III, while ξ_2^0 is given in columns 2 of Tables II and IV.

If $i\sigma_k'$ is negative any small disturbance from the unperturbed motion will die away, and this therefore is our stability criterion.

To evaluate the above integral we must assume some law for Σ_k'. Assume

$$\Sigma_k' = C\rho' = -C \text{ div } \rho_0\xi_k,$$

where C is some positive constant.

Performing a partial integration we have

$$i\sigma_k' = \frac{\gamma - 1}{2\sigma_k^{2_0}}\int_0^R \text{div } (C\rho_0\xi_k^0 + F_k') \text{ div } \xi_k^0\xi_0^2 d\xi_0. \tag{7.4}$$

Consider first the part due to the conductive term. By the usual assumptions in conduction problems we may take

$$F_k \propto \Delta(\tfrac{1}{3}aT^4),$$

and hence with the standard model

$$F_k' = -\frac{F_0}{g_0\rho_0}\{g_0\rho' + 4\Delta p_0 \cdot T'/T_0\}.$$

Taking the gas to be monatomic ($\alpha_1 = \frac{3}{2}$) in equations (2.11), etc., we find

$$4p_0T'/T_0 = \frac{1}{1 - \frac{3}{4}\beta}\{p_k' - \beta p_0\rho'/\rho_0\}. \tag{7.5}$$

Using

$$\frac{\rho'}{\rho_0} = -\text{div } \xi_k + \tfrac{3}{4}\frac{g_0\rho_0}{p_0}\xi_k, \tag{7.6}$$

we have

$$\tfrac{1}{4}\beta g_0\rho' - \beta p_0\frac{\partial}{\partial\xi_0}\left(\frac{\rho'}{\rho_0}\right) = \frac{\partial}{\partial\xi_0}(\beta p_0 \text{ div } \xi_k) + \rho_0\xi_k\left(\frac{3\beta g_0}{\xi_0} - 3\pi G\rho_0\beta\right),$$

and eliminating p_k' from (7.5), the expression for F_k' above becomes

$$F_k' = -\frac{F_0}{\rho_0 g_0(1 - \frac{3}{4}\beta)}\left\{\frac{\partial}{\partial\xi_0}(\beta p_0 \text{ div } \xi_k) + \rho_0\xi_k\left[\sigma_k^{2_0} + (4 - 3\beta)\pi G\rho_0 + \frac{3\beta g_0}{\xi_0}\right]\right\}. \tag{7.7}$$

Introducing the adiabatic approximation

$$\frac{\partial}{\partial\xi_0}(p_0 \text{ div } \xi_k) = -\frac{\rho_0\xi_k}{\gamma}\left(\sigma_k^{2_0} + \frac{4g_0}{\xi_0}\right)$$

in (7.7) we obtain by eliminating div ξ

$$F_k' = -\frac{F_0\xi_k}{g_0(1 - \frac{3}{4}\beta)}\left\{\sigma_k^{2_0}\left(1 - \frac{\beta}{\gamma}\right) + (4 - 3\beta)\pi G\rho_0 + \beta\frac{(3\gamma - 4)}{\gamma}\frac{G_0}{\xi_0}\right\}. \tag{7.8}$$

For $\alpha = 0{\cdot}1$, writing (5.8) in the form

$$\frac{\sigma_k^{2_0}\xi_0}{g_0} = bD_0$$

this becomes

$$F_k' = -\frac{F_0\xi_k}{\xi_0}\left(\cdot 86bD_0 + 3\cdot 4bD_0\frac{\pi G\rho_0}{\sigma_k^{2_0}} + 0\cdot 1\right). \tag{7.9}$$

For the fundamental a survey of Table I will convince us that the main contribution of F_k' to (7.4) arises from the term involving $\frac{dD_0}{d\xi_0}$, which is always positive. Therefore since ξ_k is positive, this contribution to the integral is negative, as div ξ_k is also positive. For the first harmonic from Table II and equation (7.9) we see that the main contribution from div F_k' div ξ_k arises from the term $\frac{d\xi_k}{d\xi_0}$, which will thus be involved to the second power in the integral. Thus the contribution will again be large and negative from the sign of (7.9).

Calling the conductive term in (7.4) $i\sigma_F'$ we may write

$$i\sigma_k' = i\sigma_F' - \frac{(\gamma-1)C}{2\sigma_k^{2_0}}\int_0^R \frac{\partial}{\partial\xi_0}(\text{div } \xi_k)\rho_0\xi_k\xi_0^2 d\xi_0$$

after performing a partial integration.

Now $(\gamma - 1)$ is positive for both the fundamental and harmonic, while for the fundamental

$$\xi_k\frac{\partial}{\partial\xi_0}(\text{div } \xi_k) = \xi_k\frac{d^2\xi_k}{d\xi_0^2}$$

is positive throughout nearly the whole star. But for the harmonic this is not so. However, a comparison with (7.9), bearing in mind the presence of the factor ρ_0, convinces us that for reasonable values of C this term is much smaller than that due to the middle term of (7.9). Thus in both cases

$$i\sigma_k' < 0 \quad \text{and} \quad \text{real.}$$

We conclude that the star is vibrationally stable, a conclusion which, as we have already indicated, is more easily reached by considering the times of decay in § 6.

Summary.

In § 2 the equations for the pulsation theory are derived in detail, on the main assumptions (1) that the oscillations are adiabatic and (2) that the change of state due to the oscillations is sufficiently small to justify the use of the linearisation process leading to the variational equations.

In § 3 two numerical solutions are given for the fundamental and first harmonic modes of vibration. These tables indicate that, for the fundamental vibration, the displacement is fairly constant throughout the main interior of the star; but for the first harmonic, while this property holds from the centre to the first node, there is a very great increase in the displacement between the first and second nodes. This leads us to anticipate that the harmonics will suffer much greater dissipation than the fundamental and consequently will not be observed.

In § 4 we calculate the theoretical period of the first harmonic and for δ Cephei the value was found to be about 1·4 days.

In § 5 the actual dissipation of energy for the different cases is tabulated, and the general conclusion is that the dissipation is greater for the first harmonic than for the fundamental mode.

In § 6, on the hypothesis of a free oscillation of the system, we find that the fundamental will decay in about 8000 years and the first harmonic in a much shorter time. Although this is so small a part of the life of a star, we might take it as an explanation for the non-appearance of the overtones in the observational data. Probably we must seek another cause for maintaining the pulsation, and by separating the opacity part of the dissipation from the remainder, we find that, with a reduction in the index of the opacity relation, free pulsations might be maintained for indefinite periods. The overtones would probably be absent from these pulsations.

In § 7 we show by considering the times of decay and by an alternative hypothesis that the star is stable when vibrating under either the fundamental or first harmonic modes.

I should like to thank Prof. Sir A. S. Eddington for his kindly interest and suggestions throughout the main part of this work.

Sidney Sussex College, Cambridge:
1933 *April* 4.

PHOTOMETRIC OBSERVATIONS OF TWILIGHT.

W. M. Smart.

(Communicated by the Director, Cambridge Observatory)

The measures described below were made with the new photo-electric photometer attached to the Sheepshanks telescope, a brief account * of which I gave at the Annual General Meeting of the Society in February last. It will be sufficient for present purposes to remark that the influence of the photo-electric current, produced by the light source which is under examination, on the needle of the Lindemann electrometer is neutralised by means of a sliding potentiometer, and that the rate at which this additional potential is applied to the needle is proportional to the intensity of the light admitted to the photo-electric cell. A scale attached to the potentiometer enables readings to be made between the zero of the scale and any particular graduation, and the time required in applying the corresponding potential to neutralise the effect of the photo-electric current is taken by a stop-watch.

The observations were made on the evenings of January 6, 9 and 25, when the atmospherical conditions were as nearly perfect as it is possible to have them in Cambridge, and they were begun when the Sun's centre was about 6° below the horizon. The telescope was pointed to the zenith

* *Observatory Magazine*, **66**, 76 (1933 March).

and kept fixed. The amount of light admitted to the cell could be varied by means of a slide with holes of different diameters, and by making a series of observations with two or more positions of the slide in succession it was a simple matter to reduce all observations to correspond to the largest hole of the slide, and in effect to the full aperture of the circular window of the photo-electric cell. At the beginning of the observations, when the intensity of twilight was considerable, a small hole was used and the corresponding times taken for 10 divisions of the sliding potentiometer scale, and towards the end of the series, when twilight was still just measurable, the full aperture of the cell was used and readings taken for one division of the scale. Such a series of observations when reduced to a common basis gives the relative amount of light from a definite area of the sky at the zenith and can be readily expressed in terms of a magnitude scale. If T_1 and T_2 are two times by the stop-watch corresponding to the full aperture of the cell and one division of the potentiometer scale, the magnitude-difference is $2{\cdot}5 \log (T_2/T_1)$. It is convenient at first to take zero magnitude to correspond to $T = 1$ second.

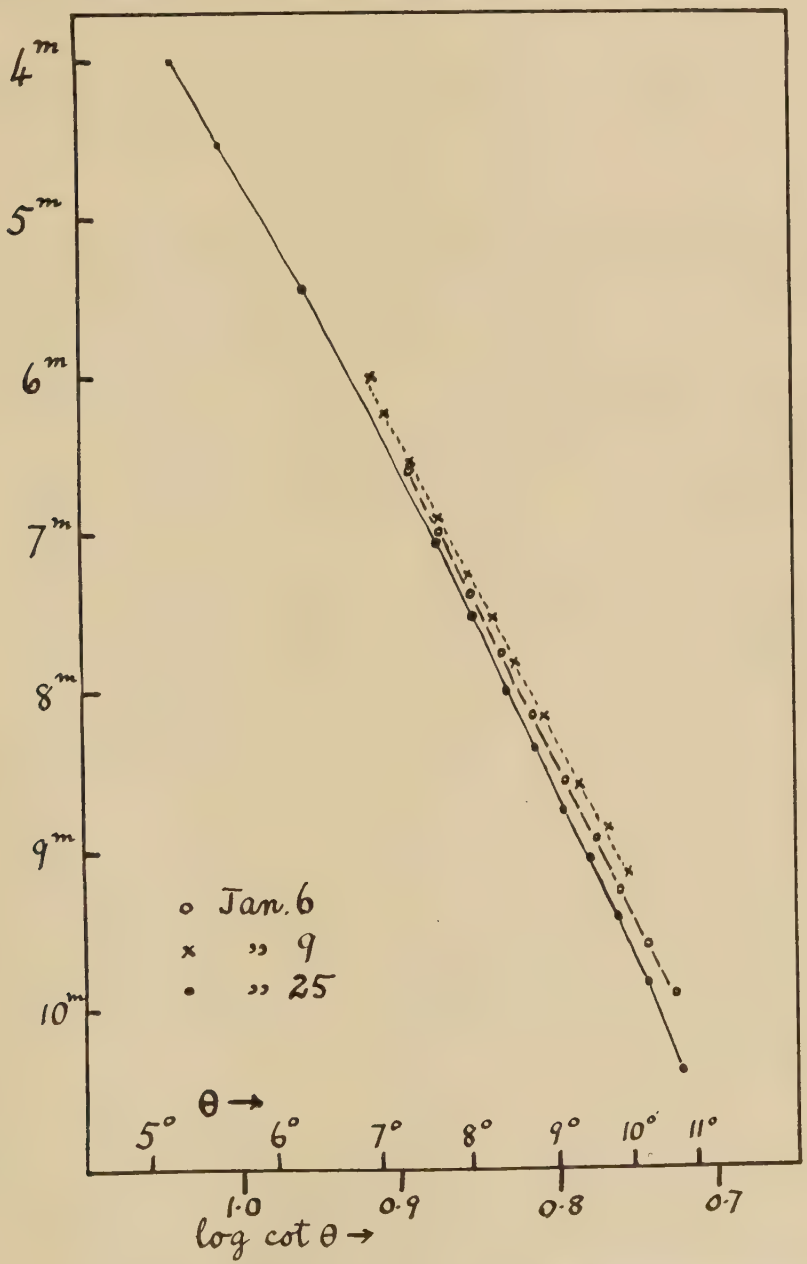
The diameter of the window of the cell which is not far from the focal plane of the telescope is 1·5 cm., and 1 cm. at the focal plane is equivalent to 5·84 minutes of arc. The area of the window thus corresponds to 60·3 square minutes of arc. The application of the factor 60·3 to the stop-watch times will thus yield the relative intensities of twilight per square minute of arc at the zenith. This is equivalent to adding 2·5 log 60·3 or 4·45 to all magnitudes on the arbitrary scale.

I have used photometric measures of δ Persei, observed in connection with our study of Algol, and made immediately at the close of the twilight observations, to provide the necessary bridge between the arbitrary magnitude scale and the usual stellar magnitude scale. Unfortunately the relation between the two scales cannot be determined with complete accuracy, since there is little chance that the state of the atmosphere remained constant in the interval of approximately an hour between the beginning of the twilight observations and the beginning of the observations of δ Persei. However, assuming that there is no conspicuous change in the atmospherical conditions, the correlation between the twilight magnitude scale and the stellar scale is simply effected. The correction to the zenith in the observations of δ Persei was applied in the usual way. The visual magnitude of δ Persei is 3·10 and its spectral type B5.

In the accompanying diagram the observations on each of the three nights, after being reduced in the manner indicated, are plotted, the stellar magnitude for 1 square minute of arc at the zenith (on the usual magnitude-scale and corresponding to type B5), against log cot θ, where θ is the angle of depression of the Sun's centre below the horizon. The corresponding values of θ are also shown. To obtain the photographic magnitude corresponding to a given value of θ, the magnitude as given by the curves should be reduced by about 0·25.

The curves are sensibly linear in the range considered. The fact that the three curves are not quite coincident is, of course, due to the somewhat

different atmospherical conditions during twilight, although for the subsequent observations of δ Persei there was remarkable uniformity in the



state of seeing on the three nights. Taking the average for the three nights we conclude that, under favourable conditions of seeing, the intensity of twilight per square minute of arc at the zenith when the Sun is 10° below the horizon is equivalent photographically to the light from a star of magnitude 9·5 approximately.

AN APPARENT LUNAR EFFECT IN TIME DETERMINATIONS AT GREENWICH AND WASHINGTON.

A. L. Loomis and H. T. Stetson.

A study in the behaviour of precision clocks, involving a close study of time-signals sent from European and American Observatories, has emphasized the fact that considerable variations exist between meridian determinations at different observatories, at times amounting to more than $0^s.050$. Thus a clock keeping a perfect rate by British time might appear to vary by this amount in a month by American time. The study by Stetson * of small changes in latitude revealing a possible correlation of displacements of the vertical with the Moon's hour-angle and declination suggested a study of the apparent longitude variation by a similar method of analysis. M. F. Jordan,† in an analysis of meridian observations at Washington and Ottawa, inferred a displacement of the vertical in an east and west direction likewise dependent upon the Moon's hour-angle and declination.

The published comparison of daily time-signals between Arlington (Washington) and Rugby (Greenwich), covering the last three years (1929–32), offered a considerable amount of material from which variations in the time determinations could be studied with a view to discovery of a lunar effect, if such existed. The fact that each station receives its own signal as well as that of the other station makes possible both the elimination of errors in sending and of the time of transmission of the signal across the Atlantic.

In the present investigation the signal from Rugby at 18^h G.C.T., corrected by its own published error, has been compared with the Arlington signal at 17^h G.C.T. similarly corrected ; the daily times taken from the Admiralty Notices for the two stations have been compared with the corresponding Arlington determinations for the two stations as published in the *U.S. Naval Observatory Time Signal Bulletin*. The treatment of the observational material is as follows :—

Let d = difference in corrected times between England and the United States.

t = time of propagation of the radio wave across the Atlantic in either direction.

a = Washington correction to NSS (Arlington) at 17^h G.C.T.

m = Greenwich correction to NSS (Arlington) at 17^h G.C.T.

b = Washington correction to GBR (Rugby) at 18^h G.C.T.

n = Greenwich correction to GBR (Rugby) at 18^h G.C.T.

* *Nature*, **123**, 127, 1929 ; also *National Research Council, Trans. Am. Geophys. Union*, 1929–30, p. 148 ; 1931, p. 45 ; 1932, p. 85.

† *Am. Astron. Society Abstracts*, **7**, 28, 1931.

Then the following equations result :—

$$d = a + t - m$$

$$d = b - t - n.$$

These equations may be solved for either d or t.

$$2d = (b - n) + (a - m), \qquad (A)$$

$$2t = (b - n) - (a - m). \qquad (B)$$

The resulting numerical data were tabulated, giving values of $2d$ for 698 calendar dates (1929–32). For obvious reasons data were omitted for a given date when either station failed in receiving satisfactory signals from both stations involved.

Errors in the recording and transmission systems should cause as large variations in $2t$ as in $2d$. A study of the curve for $2t$ shows that these variations seldom amount to more than $\pm 0^{s}{\cdot}02$, and are therefore wholly inadequate to explain the large variations in $2d$. The daily variations in $2t$ can be generally explained by assuming a random error of the order of $\pm 0^{s}{\cdot}01$ in each of the records of the four radio signals that make up each determination of $2t$.

On the hypothesis that these larger variations in $2d$ might have a lunar period, the daily values of $2d$ were entered upon cards which would allow a convenient method of sorting. On each card was also entered the Moon's hour-angle and declination for the date in question. For convenience this hour-angle was taken to be that at $17^{h}\ 30^{m}$ G.C.T. at the meridian of Washington minus 12 hours.

Experience gained in the studies of latitude variation as effected by the Moon's zenith distance suggested a division of these cards into three groups : (I) data obtained when the Moon was in declination south, $-12°$ to $-28°$; (II) when the Moon was near the equator, $-12°$ to $+12°$; and (III) when the Moon was in declination north, $+12°$ to $+28°$. The cards of each group were then arranged in the order of increasing lunar hour-angle. The first five cards of a group were taken and the mean value of $2d$ determined together with the mean lunar hour-angle of these five cards. Each succeeding five cards of the group were treated in the same manner. These data are plotted as solid dots in fig. 1. For the purpose of drawing a curve the running means of five successive solid dots were also plotted, and are shown as circles in the figure. Three curves result, showing (Curve I) the variation of $2d$ with the lunar hour-angle when the Moon was south of the equator, $-12°$ to $-28°$; (Curve II) the variations of $2d$ with the lunar hour-angle when the Moon was near the equator, $-12°$ to $+12°$; and (Curve III) the corresponding dependence when the Moon was north of the equator, $+12°$ to $+28°$.

In Curve I it will be observed that there is a steady rise in $2d$ with the increasing hour-angle of the Moon to about 14^{h}, and a fall in the values of $2d$ thereafter until hour-angle 24^{h}.

Examination of Curve III shows the opposite effect, that is, the values

of 2*d* decrease from hour-angle 0^h to 14^h, with a subsequent rise from 14^h to about 24^h. The extreme variation in magnitude of 2*d* obtained from

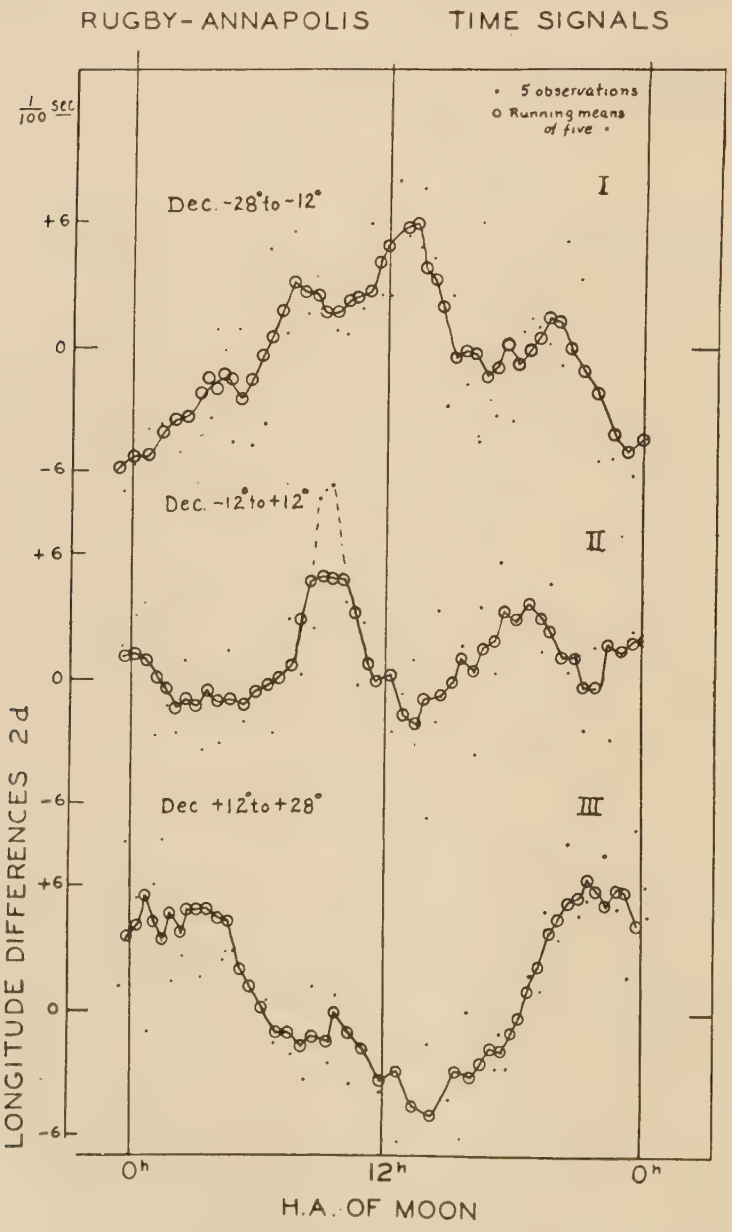


FIG. 1.—*Curve showing variation of 2d with lunar hour-angle.*
. represents five observations.
o represents running means of five observations.

the mean curve is of the order of 0^s·12 of time in both cases; for *d* this corresponds to an apparent difference in longitude 0″·9. The inversion of the phase of Curve III as compared with Curve I should be noted.

An examination of Curve II indicates two maxima near 10^h and 20^h, with a possible smaller maximum near 0^h. The range of $2d$ for the mean curve in this instance is only about half that of Curves I and III.

In a study of a series of variations in latitude with lunar hour-angle and declination,* the suggestion has been made that tidal effects in the Earth's

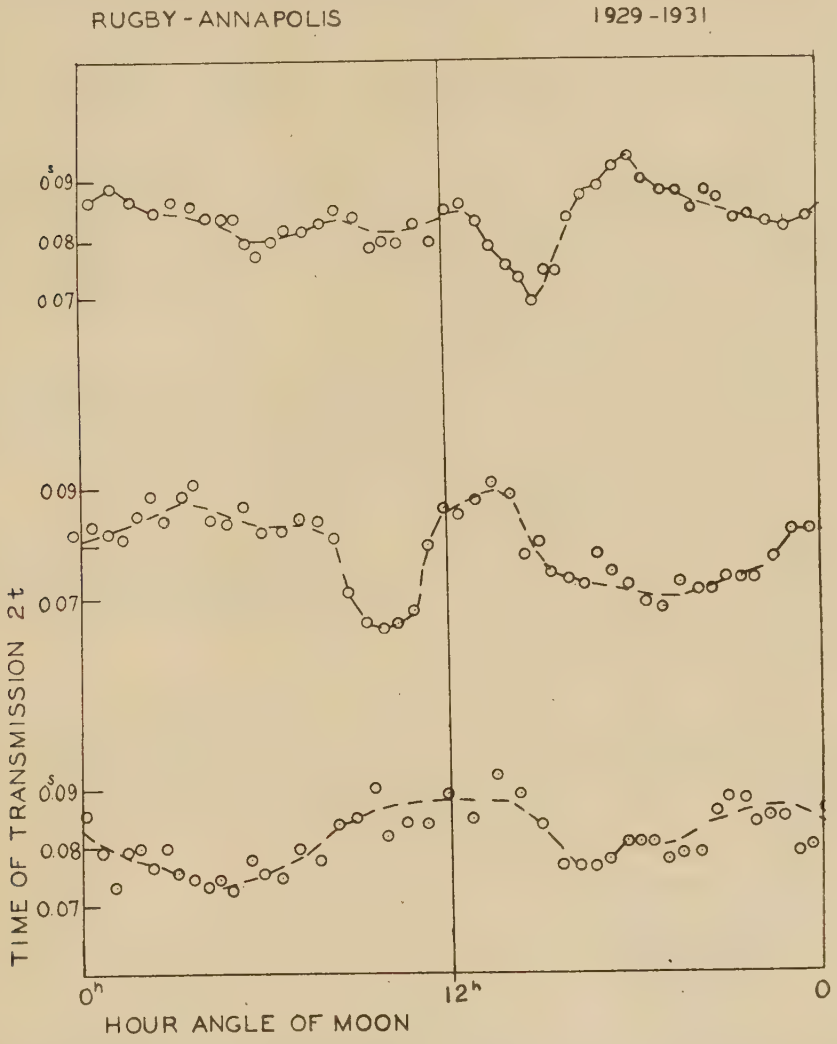


FIG. 2.—*Curve showing variation of 2t with lunar hour-angle. Upper curve, Moon south; middle curve, Moon near Equator; lower curve, Moon north.*

crust might perhaps be responsible for the displacement of the vertical by an actual small movement of, or over, the asthenosphere, which presumably corresponds to a depth approximating that of the isostatic layer. It did not appear in these latitude investigations that a sufficient change in the direction of the vertical would result from a theoretical Earth-tide to account for the

* *National Research Council, Trans. Am. Geophys. Union*, 1932, p. 85.

magnitude of the variation in question, so that an actual tangential movement of a minute amount along the Earth's surface was postulated.

Independently of any hypothesis it is interesting to note that the large changes in longitude difference in the present investigation, as represented by d, correspond to a relative shift in an east and west direction of $\frac{0''\cdot 9\,R}{206265}\cos L$, where L is 45° (the mean latitude between Greenwich and Washington) and R is the radius of the earth. This gives a numerical value for $d = 63$ feet, or a variation of about 32 feet either way from the mean positions.

While the curves in fig. 1 seem to clearly indicate that there is a lunar effect, the material so far used is not sufficiently extensive to warrant definite statements as to the origin of this effect. For this reason we are continuing this study so that we may have curves for $2d$ between other stations, particularly between Paris and London and between Paris and Washington, and between these stations and Berlin.

If this lunar effect appears in all these cases where we are considering the time differences between two stations, this effect ought also to be present in the individual time determination of each station when the clock correction at each transit is analyzed with reference to the lunar hour-angle and declination at the time of such transit.

So far in this investigation we have considered only the solution of our equations for the factor $2d$, the time of transmission of the signal t being eliminated. If we solve equation B, eliminating d, we obtain the quantity $2t$, whose average value is found to be $0^s\cdot 08$. This gives $0^s\cdot 04$ for the transmission time of the radio wave across the Atlantic.

If we treat the data for $2t$ in the same way as we have for $2d$ we obtain again a graph showing small variations of $2t$ with the hour-angle of the Moon, attaining an extreme range of about $0^s\cdot 015$ depicted in fig. 2. It should be noted that the vertical scale of fig. 2 approximately is three times that of fig. 1. It seems clear, therefore, that no such marked variations in $2t$ occur as was observed in the quantity $2d$.

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1933 *January.*

OCCULTATIONS OF STARS BY THE MOON OBSERVED AT BIRMINGHAM UNIVERSITY DURING THE YEAR 1932.

James Young.

This list is in continuation of those published in previous years. The method adopted has been described in *M.N.*, **92**, 456, 1932, and need not be repeated here.

In the reductions a correction of $0^h\cdot 0030$ has been added to the G.C.T.

when taking the Moon's position from the *Nautical Almanac*. The G.C.Ts. shown have been corrected for lag and the corrections are shown in column seven of the table. The positions of Nos. 2 and 3 have been reduced from those given in the Greenwich 1910 Catalogue. All other positions have been taken from the *Nautical Almanac*.

I am indebted to Mr. J. D. McNeile for an independent reduction of these occultations by himself and others.

No.	Star	Mag.	Phase	Date	G.C.T.	Corr.	$\sigma' - \sigma$	$\chi - \rho$
				d	h m s	s	"	° ′
1	134 B Gem	6·5	D	Feb. 18	18 19 38·8	−0·2	+1·4	+24 25
2	B.D. +27° 1362	6·9	D	Feb. 18	19 29 20·8	−0·3	+0·2	−17 57
3	B.D. +27° 1282	8·4	D	Mar. 16	21 21 43·6	−0·4	−0·2	−50 29
4	49 Aur	5·1	D	Apr. 12	18 56 24·3	−0·3	−0·8	−22 43
5	ι Vir	5·7	D	May 17	20 04 07·9	−0·2	+0·3	+32 35
6	83 Leo	6·3	D	June 11	20 31 26·8	−0·3	−1·0	−39 55
7	τ Leo	5·2	D	June 11	21 17 37·2	−0·3	−1·0	−54 14
8	49 Vir	5·2	D	June 13	20 43 51·4	−0·2	−1·0	+20 44
9	ψ Vir	5·0	D	July 10	21 24 37·6	−0·3	−0·4	+51 06
10	α Leo	1·4	D	Nov. 21	07 47 51·2	−0·2	−2·4	+41 28
11	ι Aqr	4·4	D	Dec. 3	18 23 44·6	−0·3	−1·4	+16 28

Notes.—No. 1. Mean from two observers.
,, 7. Observed through thin cloud but tolerably certain.
,, 9. Cloud cleared before D. Certain.
,, 10. Bright limb. D sudden.

University of Birmingham Observatory,
Edgbaston, Birmingham :
1933 *March* 22.

THE EQUILIBRIUM OF DISTORTED POLYTROPES.

(II) The Tidal Problem.

S. Chandrasekhar.

§ 1. In a paper * recently communicated to the Society the equilibrium of slowly rotating polytropes was studied. It was shown there that the structure of such configurations can be specified completely when two functions ψ_0 and ψ_2 related in a special way to the Emden function θ are numerically integrated. In this communication another class of distorted polytropes will be studied, namely, the class of *tidally distorted polytropes*.

§ 2. The problem which will be studied here can be formulated as follows:—

We have a gas configuration of mass M—which we shall call the *primary*—acted on by tidal forces originating from a second gas configuration of mass M'—which we shall call the *secondary*. Let the centre of gravity of the primary be taken as the origin, and let the centre of gravity of the

* "The Equilibrium of Distorted Polytropes: (I) The Rotational Problem," *M.N.*, **93**, 390, 1933. Referred to here as I.

secondary be at a distance R from the origin on the Z-axis. The potential of the secondary at great distances can be expressed in the form

$$V' = G\left(\frac{M'}{r'} + \frac{A' + B' + C' - 3I'}{2r'^3} + \ldots\right). \tag{1}$$

The term in r'^{-2} is lacking if the radius vector r' is reckoned from the centre of gravity of the secondary. A', B', C' denote the principal moments of inertia of the secondary at its centre of gravity, and I' the moment of inertia about the radius vector. These moments of inertia are of the order of magnitude $M'a'^2$ where a' is the mean radius of the secondary. Now for a spherically symmetrical distribution of matter $(A' - I')$, $(B' - I')$, $(C' - I')$ are all identically zero, and since, as is well known, the distortions arising from tidal influences is of the third order in the ratio of the radius of configuration to the distance between the components, we have the result that $(A' - I')$, $(B' - I')$, $(C' - I')$ for the secondary distorted by the tidal field due to the primary * is of the order $M'a'^2(a'R^{-1})^3$. Hence the second term in (1) is of the order of magnitude

$$G' \cdot \left(\frac{M'}{a'}\right) \cdot \left(\frac{a'}{R}\right)^6. \tag{2}$$

We shall neglect quantities of this order, *i.e. we shall consistently carry out our approximation to that order of accuracy in which the effects of the secondary on the primary are the same as that of a mass point.*†

The potential V' at a point (x, y, z) in the neighbourhood of the primary is therefore

$$V' = \frac{GM'}{r'} = \frac{GM'}{R}\left(1 - 2\frac{z}{r}\left(\frac{r}{R}\right) + \frac{r^2}{R^2}\right)^{-\frac{1}{2}}, \tag{3}$$

or writing $z/r = \cos\theta = \mu$ and expanding (3) in terms of the Legendre polynomials, $P_j(\mu)$ we have

$$V' = \frac{GM'}{R}\sum_{j=1}^{\infty}\left(\frac{r}{R}\right)^j P_j(\mu) + \text{constant}. \tag{4}$$

Consistently with our scheme of approximation where we neglect quantities of the order (2) we have to retain only the first four terms in the above summation, *i.e.*

$$V' = \frac{GM'}{R^2} r\cos\theta + \frac{GM'}{R}\sum_{j=2}^{4}\left(\frac{r}{R}\right)^j P_j(\mu) + \text{constant}. \tag{5}$$

The first term on the right gives a uniform field of force of intensity M'/R^2, which produces a Newtonian acceleration M'/R^2 in the primary. We can

* Which means that we are *not* considering an *arbitrary* secondary but one which (like the primary) is also in equilibrium under the combined effect of its own gravitation and the tidal forces due to the *primary*.

† Another point in this connection may be explicitly stated. The next term in the expansion (1) is of order involving r'^{-5} (*i.e.*) R^{-5}. This term is zero for a sphere. Since the distortions are of the order $(a'/R)^3$ we see that we can neglect these terms, consistently with our scheme of approximation. (I am indebted to the referee of my paper for this remark.)

neutralise this term by supposing the axes of reference to move with this acceleration. The centre of gravity of the primary will then always be at the origin.*

We are thus left with a *tide-generating potential*

$$V_T = \frac{GM'}{R}\sum_{j=2}^{4}\left(\frac{r}{R}\right)^j P_j(\mu) + \text{constant}, \tag{6}$$

and we have to consider the effect of such a tidal potential on a gas configuration in which the density distribution is originally polytropic. In other words, we are given that in the primary we have a relation of the type

$$P = K\rho^{1+\frac{1}{n}}, \tag{7}$$

where P is the total pressure and ρ is the density. In the absence of the tidal field V_T the density distribution is governed by the Emden function θ of index $-n$. *The tidal problem is to determine the shape and the density distribution in this configuration when distorted by a tidal field of the form* V_T *and explicitly relate its structure with the Emden function describing the undistorted state.*

Formulated in this way the problem and the standpoint taken here are different from the earlier investigations on the effect of tidal forces on " compressible stellar masses."

The problem considered here does not seem to have been attempted before, even for the special case $n = 3$.

§ 3. *The Equations of the Problem.*—The equations of mechanical equilibrium are

$$\text{grad } P = \rho \text{ grad } (V + V_T), \tag{8}$$

$$\text{div grad } V = -4\pi G\rho, \tag{9}$$

$$\text{div grad } V_T = 0. \tag{10}$$

From (8), (9) and (10) we deduce that

$$\text{div}\left(\frac{1}{\rho}\text{grad } P\right) = -4\pi G\rho. \tag{11}$$

Changing to polar co-ordinates r, μ, ϕ and omitting the term in ϕ from symmetry and introducing the Θ and ξ variables defined by

$$r = \alpha\xi = \left[\frac{(n+1)K\lambda^{\frac{1}{n}-1}}{4\pi G}\right]^{\frac{1}{2}}\xi, \tag{12}$$

$$\rho = \lambda\Theta^n; \qquad P = K\lambda^{1+\frac{1}{n}}\Theta^{n+1}, \tag{13}$$

we find that (11) reduces to

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right) = -\Theta^n. \tag{14}$$

* Compare J. H. Jeans, *Problems of Stellar Dynamics and Cosmogony* (Cambridge, 1919), p. 43, § 47.

§ 4. *Solution.*—We assume a solution of the form

$$\Theta = \theta + \Psi(\xi, \mu), \tag{15}$$

where θ is Emden's function of index n and Ψ is a function numerically small compared with θ, so that we can neglect quantities in Ψ^2 and higher orders. Substituting (15) in (14) and neglecting terms in Ψ^2, we have for the differential equation defining Ψ:

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Psi}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right)=-n\theta^{n-1}\Psi. \tag{16}$$

Now, we take for Ψ the following form :

$$\Psi=\sum_{j=1}^{\infty}\psi_j(\xi)A_jP_j(\mu), \tag{16'}$$

where A_j's are at present arbitrary, to be determined later. In the case of the tidal problem there is no purely radial function analogous to $\psi_0(\xi)$ which appeared in the rotational problem, for it is clear that "$\psi_0(\xi)$" refers to an expansion of the configuration as a whole (*cf.* I, § 7), and such an expansion is not possible in the tidal problem. (The formal justification is given later. See footnote on page 453.)

Substitution of (16') in (16) yields for ψ_j the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\left(\frac{j(j+1)}{\xi^2}-n\theta^{n-1}\right)\psi_j, \qquad j=1, 2, 3, \ldots \tag{17}$$

For the determination of the coefficients A_j we have to calculate the potential for reasons stated in I. The calculation proceeds exactly as in I (§ 6), and we shall not go into the details. The result is that

$$V=(n+1)K\lambda^{\frac{1}{n}}\left[\Theta+\sum_{j=1}^{\infty}B_j\xi^jP_j(\mu)\right], \tag{18}$$

where the B_j's are at present arbitrary. To determine first the B_j's, we have to substitute the above expression for V in the equation of mechanical equilibrium

$$\frac{\partial P}{\partial\xi}=\rho\left(\frac{\partial V}{\partial\xi}+\frac{\partial V_T}{\partial\xi}\right), \tag{19}$$

where the tidal potential V_T (given by (6)) expressed in the ξ-variable takes the form

$$V_T=\frac{GM'}{R}\sum_{2}^{4}\left(\frac{a}{R}\right)^jP_j(\mu)\,.\,\xi^j. \tag{20}$$

On going through the algebra and equating the coefficients of the various Legendre-polynomials we find that

$$B_j=0 \quad \text{if} \quad j\neq 2, 3, \text{ or } 4,$$

and

$$B_j=-\frac{GM'}{R}\frac{1}{(n+1)K\lambda^{\frac{1}{n}}}\left(\frac{a}{R}\right)^j, \qquad j=2, 3, 4. \tag{21}$$

We can simplify the above expression for B_j by using the equation expressing the mass (see equation (31) below), namely,

$$M = 4\pi\alpha^3\xi_1^2\lambda \mid \theta_1' \mid. \tag{22}$$

(21) now simplifies to

$$B_j = -\frac{M'}{M}\xi_1^2 \mid \theta_1' \mid \left(\frac{\alpha}{R}\right)^{j+1}, \qquad j = 2, 3, 4. \tag{23}$$

Hence our expression for the potential is by (18)

$$V = (n+1)K\lambda^{\frac{1}{n}}\left[\Theta - \sum_{j=2}^{4}\frac{M'}{M}\xi_1^2 \mid \theta_1' \mid \left(\frac{\alpha}{R}\right)^{j+1}\xi^j P_j(\mu)\right]. \tag{18'}$$

Now the above expression for V and its derivative will have to be continuous on a sphere of radius ξ_1 (the first zero of Emden's function θ) with an expression of the type,

$$V_{\text{external}} = (n+1)K\lambda^{\frac{1}{n}}\left[\frac{C_0}{\xi} + \sum_{j=1}^{\infty}\frac{C_j}{\xi^{j+1}}P_j(\mu)\right]. \tag{24}$$

This will determine the A_j's. We find that

$$A_j = 0 \quad \text{if} \quad j \neq 2, 3 \text{ or } 4, \tag{25}$$

and for $j = 2$, 3 and 4 we have

$$A_j = (2j+1)\xi_1^{j+2} \cdot \frac{M'}{M} \cdot \mid \theta_1' \mid \cdot \left(\frac{\alpha}{R}\right)^{j+1} \cdot \frac{1}{(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)}. \tag{26}$$

Hence the solution to the problem * is expressed by

$$\Theta = \theta + \xi_1 \mid \theta_1' \mid \cdot \nu^3 \cdot \frac{M'}{M}\sum_{j=2}^{4}(2j+1)\nu^{j-2} \cdot \frac{\psi_j(\xi)P_j(\mu)}{(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)}, \tag{27}$$

where

$$\nu = \frac{\alpha\xi_1}{R}. \tag{28}$$

ν has the simple meaning, that it is the *ratio of the radius of the undistorted configuration to the distance between the centre of gravities of the two components.* On introducing the quantities Δ_j defined by

$$\Delta_j = \frac{(2j+1)\psi_j(\xi_1)}{(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)}, \qquad j = 2, 3, 4, \tag{29}$$

we can rewrite (27) in the form

$$\Theta = \theta + \xi_1 \mid \theta_1' \mid \cdot \frac{M'}{M}\sum_{j=2}^{4}\nu^{j+1} \cdot \Delta_j \cdot \frac{\psi_j(\xi)}{\psi_j(\xi_1)} \cdot P_j(\mu). \tag{30}$$

---

* The reason why we cannot have any purely radial function can be seen as follows: The radial function can be expressed as $A_0\psi_0(\xi)$ where $\psi_0(\xi)$ will satisfy the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0,$$

but A_0 will be left undetermined even when all the conditions of the problem are satisfied. Going to the limit $R \to \infty$, we see that $A_0 = 0$.

(30) *shows that consistently with our approximation in treating the secondary as a mass point, we can solve the tidal problem correctly to the fifth order in* ν.

§ 5. *Mass and Volume Relations.*—The mass is given by

$$M = 2\pi \iint \rho r^2 dr d\mu.$$

Remembering that

$$\int_{-1}^{+1} P_j(\mu) d\mu = 0,$$

we see that the terms in $P_j(\mu)$ in (30) do not contribute to the mass to the order of accuracy with which we are concerned. Hence (*cf.* I, equation (41)).

$$M = -4\pi\alpha^3 \lambda \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_1. \tag{31}$$

Similarly the volume N of the configuration is given by

$$N = \frac{4\pi}{3}\alpha^3 \xi_1^3, \tag{32}$$

the "ellipticity-terms" again not contributing to the volume. The relation between the mean and the central density is given by

$$\rho_{\text{mean}} = 3\xi_1 \cdot |\theta_1'|^{-1} \cdot \rho_{\text{central}}, \tag{33}$$

the same relation as for undistorted polytropes. Relations (31), (32) and (33) express the following fact :—

*If we consider a gas sphere and slowly bring towards it a tidally disturbing secondary, the configuration then gets distorted in shape, but in such a way that the volume and the central density remain constant.**

§ 6. *External Shape.*—The boundary ξ_0 is given by $\Theta = 0$ or by (30) we have for the equation of the boundary

$$\xi_0 = \xi_1 \left\{ 1 + \frac{M'}{M} \sum_{j=2}^{4} \Delta_j \cdot \nu^{j+1} \cdot P_j(\mu) \right\}, \tag{34}$$

or differently

$$\frac{\xi_0 - \xi_1}{\xi_1} = \frac{M'}{M} \sum_{j=2}^{4} \Delta_j \cdot \nu^{j+1} \cdot P_j(\mu). \tag{35}$$

It may be noted here that

$$\left.\begin{aligned} P_2(\mu) &= \tfrac{1}{2}(3\mu^2 - 1) \\ P_3(\mu) &= \tfrac{1}{2}(5\mu^3 - 3\mu) \\ P_4(\mu) &= \tfrac{1}{8}(35\mu^4 - 30\mu^2 + 3) \end{aligned}\right\}. \tag{36}$$

* This theorem shows that the conjecture made in I (§ 13, remark 3) is perhaps not correct. For it is known that in the tidal problem we also meet with two different types of break-up analogous to the two types of break-up met with in the rotational problem. Actually the fact that the effect of rotation is to decrease the central density when $n < 3$ and increase it when $n > 3$ appears more to have a bearing on questions of "*stability*" than on questions of "*break-up.*" A more specific discussion on this point will be attempted in a separate paper.

The surface of the configuration as expressed by (34) is symmetrical with respect to the Z-axis but unsymmetrical with respect to the other two axes. There is an elongation at either of the poles ($\theta = 0$ or π) but of different amounts. The elongation at the pole nearer the secondary is greater than that at the opposite end. At the equator, on the other hand, there is a *uniform contraction.* We shall denote by σ_{+1}, σ_{-1} and σ_0 the fractional elongation at the pole $\theta = 0 (\mu = +1)$, the fractional elongation at the opposite pole $\theta = \pi (\mu = -1)$ and the fractional *contraction* at the equator $\theta = \pi/2 (\mu = 0)$ respectively. Remembering that $P_j(\mu)$ $(j = 2, 3, 4)$ takes for $\mu = 0$, $+1$ and -1 the values

μ	0	$+1$	-1
$P_2(\mu)$	$-\frac{1}{2}$	$+1$	$+1$
$P_3(\mu)$	0	$+1$	-1
$P_4(\mu)$	$\frac{3}{8}$	$+1$	$+1$

given in the table, we derive from (35) the following explicit relations for σ_{+1}, σ_{-1} and σ_0 :—

$$\left.\begin{aligned} \sigma_{+1} &= \frac{M'}{M}(\Delta_2\nu^3 + \Delta_3\nu^4 + \Delta_4\nu^5) \\ \sigma_{-1} &= \frac{M'}{M}(\Delta_2\nu^3 - \Delta_3\nu^4 + \Delta_4\nu^5) \\ \sigma_0 &= \frac{M'}{M}(\tfrac{1}{2}\Delta_2\nu^3 - \tfrac{3}{8}\Delta_4\nu^5) \end{aligned}\right\}. \tag{37}$$

We note that

$$\sigma_{+1} - \sigma_{-1} = 2\frac{M'}{M} \cdot \Delta_3\nu^4. \tag{38}$$

Further, we see from (37) that

$$\frac{\sigma_{+1}}{\sigma_0} = \frac{\sigma_{-1}}{\sigma_0} + O(\nu) = 2 + O(\nu), \tag{39}$$

or in words (39) expresses the following fact :—

The fractional elongations at the two poles are to a first approximation equal and are to the same approximation just twice the uniform fractional contraction at the equator. Further, this fact is true for all polytropes.

The above is, of course, nothing new in principle and is primarily a numerical consequence of the values of $P_2(\mu)$ for $\mu = 0$ and 1. Indeed (39) must be true for all density distributions in the configuration, because in the tidal problem there is no expansion of the configuration as a whole, and to a first order the equation of the boundary must be of the following form when slightly distorted by a tidal field:—

$$\xi_0 = \xi_1 + \text{constant} \cdot P_2(\mu). \tag{40}$$

Attention may here be drawn to the fact that in the rotational problem (as against the tidal problem) the ratio of the elongation at the equator to the contractions at the poles is rather sensitive to the polytropic index, increasing from 1·5 for $n=1$ to 13·3 for $n=4$.* This difference between the two problems arises because in the rotational problem there is a general expansion of the configuration as a whole and this expansion is rather sensitive to the undisturbed density distribution with which we start.

Again, from (35) we see that *if we neglect terms in* ν^4, *the configurations are spheroids with a small ellipticity* (*or oblateness*) ϵ *given by*

$$\epsilon = 1{\cdot}5 \cdot \frac{M'}{M} \cdot \Delta_2 \nu^3. \tag{41}$$

§ 7. *Numerical Results.*—The structure of the tidally distorted polytrope will thus be specified completely when the functions ψ_2, ψ_3 and ψ_4 are known. The functions ψ_2 for indices of n equal to 1, 1·5, 2, 3, 4 have been already tabulated in I (Appendix). The functions ψ_3 and ψ_4 have also been integrated for the same values of n and are tabulated in an appendix to this paper. At the bottom of each table the values of ψ_3 and ψ_4 and their derivatives at ξ_1, the first zero of θ, are given.

The series expansions with which the numerical integrations were started may be noted here.

$$\psi_3 = \xi^3\left[1 - \frac{n}{18}\xi^2 + \frac{n(4n-3)}{18\cdot 44}\xi^4 - \frac{n(434n^2-749n+330)}{18\cdot 44\cdot 1170}\xi^6 + \ldots\right], \tag{42}$$

$$\psi_4 = \xi^4\left[1 - \frac{n}{22}\xi^2 + \frac{n(14n-11)}{66\cdot 52}\xi^4 - \frac{n(1744n^2-3129n+1430)}{1350\cdot 66\cdot 52}\xi^6 + \ldots\right]. \tag{43}$$

In Table VI the values of ψ_2, ψ_3, ψ_4 and also their derivatives at ξ_1 are collected.

Table VI

n	1	1·5	2	3	4
ξ_1	3·14159	3·6538	4·3529	6·8968	14·9715
$\psi_2(\xi_1)$	4·55940	4·7820	5·6431	11·2780	46·5444
$\psi_2'(\xi_1)$	0·42060	1·1495	1·7559	3·0409	6·1766
Δ_2	1·51985	1·2892	1·1482	1·0289	1·00267
$\psi_3(\xi_1)$	17·3743	23·369	35·572	124·814	1177·24
$\psi_3'(\xi_1)$	9·7959	14·828	21·852	53·366	235·65
Δ_3	1·2129	1·1079	1·0488	1·00736	1·00047
$\psi_4(\xi_1)$	61·1506	100·088	188·075	1086·30	22684·42
$\psi_4'(\xi_1)$	59·023	96·44	164·06	626·08	6058·63
Δ_4	1·1205	1·0562	1·0231	1·00281	1·00014
$\xi_1 \cdot \lvert\theta_1'\rvert$	1·0000	0·74283	0·55391	0·29263	0·12004

* Compare I, § 13, remark (2).

The values of Δ_2, Δ_3, Δ_4 (which occur in the equations (30) and (34) expressing Θ and the boundary respectively) are also included in heavy type. In the last row the values of $\xi_1 \mid \theta_1' \mid$ are also inserted so that the exact numerical forms of (30) and (34) for different values of n can be read out at once.

It is of interest to note the inequality

$$\Delta_2 > \Delta_3 > \Delta_4, \tag{44}$$

which is seen to exist for all indices.

It is a rather curious fact that Δ_2, Δ_3, Δ_4 (the coefficients of the third, fourth and the fifth powers of ν in the equation for the boundary) tend rapidly to unity as n approaches 5. As a consequence of this we can predict at once that the *expression*

$$\xi_0 \equiv \xi_1 \left\{ 1 + \frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot P_j(\mu) \right\}, \tag{45}$$

will represent to a high degree of approximation the equation of the boundary of tidally distorted polytropes with $5 > n > 4$.

§ 8. *The Formation of a "Furrow" on the Boundary.*—There is still another feature of the boundary that remains to be pointed out.

If we take the equation of the boundary (34) then the equation

$$\frac{\partial \xi_0}{\partial \mu} = 0 \tag{46}$$

has a solution apart from $\mu = \pm 1$ or 0. Equation (46), when written out fully, takes the form

$$3\Delta_2 \cos\theta + \Delta_3 \cdot \left(\tfrac{15}{2}\cos^2\theta - \tfrac{3}{2}\right) \cdot \nu + \Delta_4\left(\tfrac{35}{2}\cos^3\theta - \tfrac{15}{2}\cos^2\theta\right)\nu^2 = 0. \tag{47}$$

An approximate solution can easily be obtained by setting $\theta = \frac{\pi}{2} - \chi$ and neglecting terms in $\sin^2\chi \cdot \nu$, $\sin\chi \cdot \nu^2$ and higher orders. We find that

$$\chi = \tfrac{1}{2} \cdot \frac{\Delta_3}{\Delta_2} \cdot \nu + O(\nu^3). \tag{48}$$

This angle χ corresponds to a "*furrow*" having developed on the surface. The following table gives this angle (in radians) :—

n	1	1·5	2	3	4	$n > 4$
χ	0·3995ν	0·4297ν	0·4567ν	0·4895ν	0·4989ν	0·5ν

This concludes the formal analysis of the simple tidal problem. But as both the rotational and the simple tidal problems are very intimately related to the "double-star" problem it is better to study this problem before attempting any detailed discussion with references to actual physical applications.

APPENDIX

Solutions of the Differential Equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_3}{d\xi}\right)=\left(-n\theta^{n-1}+\frac{12}{\xi^2}\right)\psi_3$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_4}{d\xi}\right)=\left(-n\theta^{n-1}+\frac{20}{\xi^2}\right)\psi_4$$

for n = 1, 1·5, 2, 3 *and* 4.

The values of the Emden functions were taken from Comrie and Sadler's tables. Only as many decimal figures are retained as are regarded to be reliable.

TABLE I

Index $n=1$

ξ	ψ_3	ψ_4
0·	0·	0·
0·2	0·0079822	0·0015971
0·4	0·063362	0·025415
0·6	0·21171	0·12750
0·8	0·49406	0·39783
1·0	0·94569	0·95541
1·2	1·5942	1·94152
1·4	2·4582	3·5118
1·6	3·5463	5·8272
1·8	4·8566	9·0440
2·0	6·3760	13·3043
2·2	8·0813	18·7252
2·4	9·9390	25·3918
2·6	11·9066	33·3475
2·8	13·9343	42·5878
3·0	15·9660	53·0560
3·2	17·9419	64·6402
3·4	19·8003	77·1728

$\xi_1 = 3{\cdot}14159$

$\psi_3(\xi_1) = 17{\cdot}3743$, $\psi_4(\xi_1) = 61{\cdot}1506$,

$\psi_3'(\xi_1) = 9{\cdot}7959$, $\psi_4'(\xi_1) = 59{\cdot}023$.

TABLE II—*Index* $n=1\cdot5$

ξ	ψ_3	ψ_4
0·	0·	0·
0·2	0·0079732	0·0015957
0·4	0·063156	0·025323
0·6	0·20968	0·12649
0·8	0·48584	0·39244
1·0	0·92205	0·93596
1·2	1·5395	1·8874
1·4	2·3501	3·3867
1·6	3·3565	5·5758
1·8	4·5543	8·5928
2·0	5·9334	12·5691
2·2	7·4806	17·6281
2·4	9·1825	23·8883
2·6	11·0281	31·4685
2·8	13·0112	40·497
3·0	15·1341	51·127
3·2	17·4109	63·552
3·4	19·8730	78·045
3·6	22·5836	95·025

$\xi_1=3\cdot6538$

$\psi_3(\xi_1)=23\cdot3689,\qquad \psi_4(\xi_1)=100\cdot088,$

$\psi_3'(\xi_1)=14\cdot828,\qquad \psi_4'(\xi_1)=96\cdot44.$

TABLE III—*Index* $n=2$

ξ	ψ_3	ψ_4
0·	0·	0·
0·2	0·0079764	0·0015942
0·4	0·062883	0·025234
0·6	0·20770	0·12552
0·8	0·47808	0·38734
1·0	0·90039	0·91815
1·2	1·49116	1·83962
1·4	2·2580	3·2801
1·6	3·2015	5·3701
1·8	4·3178	8·2392
2·0	5·6012	12·0163
2·2	7·0465	16·8328
2·4	8·6512	22·8270
2·6	10·4167	30·1502
2·8	12·3497	39·0073
3·0	14·4621	49·5550
3·2	16·7722	62·023
3·4	19·3044	76·681
3·6	22·0894	93·842
3·8	25·1646	113·877
4·0	28·5745	137·219
4·2	32·3714	164·378

$\xi_1=4\cdot3529$

$\psi_3(\xi_1)=35\cdot572,\qquad \psi_4(\xi_1)=188\cdot075,$

$\psi_3'(\xi_1)=21\cdot852,\qquad \psi_4'(\xi_1)=164\cdot06.$

TABLE IV

Index $n=3$

ξ	ψ_3	ψ_4
0·	0·	0·
0·2	0·0079471	0·0015913
0·4	0·06235	0·025059
0·6	0·20395	0·12367
0·8	0·46376	0·37791
1·0	0·86199	0·88643
1·2	1·4092	1·75815
1·4	2·1091	3·1073
1·6	2·9623	5·0521
1·8	3·9686	7·7159
2·0	5·1296	11·2290
2·2	6·4491	15·7440
2·4	7·9350	21·4078
2·6	9·5973	28·3959
2·8	11·4492	36·9024
3·0	13·5063	47·1426
3·2	15·7859	59·3540
3·4	18·3073	73·797
3·6	21·0906	90·755
3·8	24·1571	110·537
4·0	27·5291	133·473
4·2	31·2290	159·921
4·4	35·280	190·259
4·6	39·706	224·894
4·8	44·529	264·251
5·0	49·774	308·781
5·2	55·465	358·957
5·4	61·624	415·272
5·6	68·274	478·244
5·8	75·439	548·41
6·0	83·140	626·31
6·2	91·399	712·53
6·4	100·236	807·64
6·6	109·670	912·24
6·8	119·721	1026·94
7·0	130·402	1152·35

$\xi_1 = 6{\cdot}8968$

$\psi_3(\xi_1) = 124{\cdot}814, \qquad \psi_4(\xi_1) = 1086{\cdot}30,$

$\psi_3'(\xi_1) = 53{\cdot}366, \qquad \psi_4'(\xi_1) = 626{\cdot}08.$

TABLE V—*Index* $n=4$

ξ	ψ_3	ψ_4
0·	0·	0·
0·2	0·00797297	0·00158850
0·4	0·061827	0·024889
0·6	0·20037	0·12192
0·8	0·45091	0·36945
1·0	0·82912	0·85931
1·2	1·34245	1·69182
1·4	1·9938	2·9731
1·6	2·7854	4·8160
1·8	3·7209	7·3423
2·0	4·8062	10·6857
2·2	6·0499	14·9943
2·4	7·4632	20·4322
2·6	9·0586	27·1805
2·8	10·8508	35·4385
3·0	12·8550	45·4237
3·2	15·0876	57·3723
3·4	17·5653	71·5386
3·6	20·3053	88·1959
3·8	23·3249	107·636
4·0	26·6418	130·168
4·2	30·2736	156·122
4·4	34·2383	185·844
4·6	38·5536	219·700
4·8	43·2376	258·074
5·0	48·3080	301·369
5·2	53·7829	350·003
5·6	66·0178	465·068
6·0	80·0856	607·00
6·4	96·1293	779·81
6·8	114·292	987·82
7·2	134·716	1235·61
7·6	157·542	1528·06
8·0	182·914	1870·35
8·4	210·978	2267·90
8·8	241·849	2726·46
9·2	275·693	3252·02
9·6	312·630	3850·9
10·0	352·829	4529·6
10·4	396·395	5295·0
10·8	443·475	6154·2
11·2	494·207	7114·5
11·6	548·724	8183·7
12·0	607·163	9369·6
12·4	669·657	10680·5
12·8	736·341	12124·8
13·2	807·349	13711·2
13·6	882·813	15448·8
14·0	962·867	17346·8
14·4	1047·646	19414·9
14·8	1137·281	21662·8

$\xi_1 = 14{\cdot}9715$

$\psi_3(\xi_1) = 1177{\cdot}244, \qquad \psi_4(\xi_1) = 22684{\cdot}42,$

$\psi_3'(\xi_1) = 235{\cdot}648, \qquad \psi_4'(\xi_1) = 6058{\cdot}63.$

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THE EQUILIBRIUM OF DISTORTED POLYTROPES.

(III) The Double-star Problem.

S. Chandrasekhar.

§ 1. In two communications * the equilibrium of two types of distorted polytropes was studied. In this paper a third class of distorted polytropes will be considered, namely, the components of "double-star" systems which are both tidally and rotationally distorted. As will be shown in the following, no further quadratures are necessary to solve this problem.

The double-star problem can be formulated as follows :—

We have a system of two gas configurations revolving about their common centre of gravity. We assume that the ratio of the mean radii of the components to their distance apart is so small that we can neglect quantities of the sixth order in this ratio, since this enables one to consider (as shown in II, § 4) one of the configurations as a mass point when studying the equilibrium of the other. We fix our attention on one of the components of our "double-star" system. It is given that in this the total pressure P is related to the density ρ by means of the relation

$$P = K\rho^{1+\frac{1}{n}}. \tag{1}$$

If we disregard the tidal influence arising from the secondary and also the effects of rotation, the structure of the configuration is completely specified when the associated Emden function of index n is known. We now take into account the effects of distortion due to the tidal and the rotational forces. *The problem is to determine the shape and the density distribution in such tidally and rotationally distorted polytropes and explicitly to relate the structure of such configurations with the Emden functions which describe the polytropic state.*

Substantially the case $n=3$ has already been treated by von Zeipel,† but, as will be seen, the solution given here is a direct consequence of the methods developed in I and II.

§ 2. *Equations of the Problem.*—We take as the origin of our system of co-ordinates the centre of gravity of the primary. The Z-axis is parallel to the axis of rotation, and the X-axis is directed towards the centre of gravity of the secondary. Let the masses of the primary and the secondary be respectively M and M'. Let R be the distance between the centres of gravity. By Kepler's third law we have the relation

$$\omega^2 = \frac{G(M+M')}{R^3}(1+\epsilon), \tag{2}$$

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* "The Equilibrium of Distorted Polytropes: (I) The Rotational Problem," *M.N.*, 93, 390, 1933 ; "The Equilibrium of Distorted Polytropes: (II) The Tidal Problem," the previous paper. These two are referred to as I and II in the sequel.

† H. von Zeipel, *M.N.*, 84, 702, 1924.

where ϵ is a quantity of the order of magnitude $(\bar{a}/R)^8$,* where $\bar{a}$ represents the mean radii of the components. As we are going to carry our approximation only to the fifth order in quantities of this magnitude, we shall accordingly use the relation

$$\omega^2 = \frac{G(M+M')}{R^3}. \tag{2'}$$

From (2') it is clear that neglecting quantities which are of the sixth order in $(\bar{a}/R)$ is equivalent to neglecting quantities of the second order in ω^2.

Let V be the inner attraction potential in the primary and V' the outer attraction potential due to the secondary. The equations of mechanical equilibrium are

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= \rho\left\{\frac{\partial}{\partial x}(V+V') + \omega^2 x - \frac{M'}{M+M'}R\omega^2\right\} \\ \frac{\partial P}{\partial y} &= \rho\left\{\frac{\partial}{\partial y}(V+V') + \omega^2 y\right\} \\ \frac{\partial P}{\partial z} &= \rho\left\{\frac{\partial}{\partial z}(V+V')\right\} \end{aligned}\right\}, \tag{3}$$

where P is related to ρ by (1). Also V and V' satisfy respectively Poisson's and Laplace's equations :

$$\nabla^2 V = -4\pi G\rho\,; \qquad \nabla^2 V' = 0. \tag{4}$$

From (3) and (4) we deduce that

$$\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} P\right) = -4\pi G\rho + 2\omega^2. \tag{5}$$

Changing over to polar co-ordinates and introducing the ξ and Θ variables as defined in I and II we find that (5) reduces to

$$\nabla^2\Theta = -\Theta^n + v, \tag{6}$$

where

$$v = \frac{\omega^2}{2\pi G\lambda}. \tag{7}$$

§ 3. We seek a solution of (6) in the form

$$\Theta = \theta + v\Psi + v^2\Phi + \ldots, \tag{8}$$

where θ is Emden's function of index n. The differential equation defining Ψ is easily found to be

$$\nabla^2\Psi = -n\theta^{n-1}\Psi + 1. \tag{9}$$

We take for Ψ the following form:

$$\Psi = \psi_0(\xi) + \sum_{j=1}^{\infty}\psi_j(\xi)S_j(\mu, \phi), \tag{10}$$

* Compare G. H. Darwin, *Scientific Papers*, 3, § 15, p. 479.

where $S_j(\mu, \phi)$, $(\mu = \cos\theta)$ is at present an arbitrary surface-harmonic of degree j satisfying the differential equation

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial S_j}{\partial\mu}\right)+\frac{1}{1-\mu^2}\frac{\partial^2 S_j}{\partial\phi^2}+j(j+1)S_j=0. \tag{11}$$

Substituting (10) in (9) and using (11) and equating the coefficients of the successive orders of the surface harmonics we find that the differential equations defining the radial functions $\psi_0(\xi)$ and $\psi_j(\xi)$ are

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)=-n\theta^{n-1}\psi_0+1, \tag{12}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\left(-n\theta^{n-1}+\frac{j(j+1)}{\xi^2}\right)\psi_j, \qquad j=1, 2, 3, \ldots \tag{13}$$

To determine the precise forms of the surface-harmonics S_j we have, as usual, to calculate the potential. The result is

$$V=(n+1)\,.\,K\,.\,\lambda^{\frac{1}{n}}\left[\Theta+v\left\{\sum_{j=1}^{\infty}\xi^j S_j'(\mu,\phi)-\tfrac{1}{6}\xi^2\right\}\right], \tag{14}$$

where S_j''s are again surface harmonics of the successive orders, arbitrary at present and different from the S_j's introduced in (10). To determine the precise forms of these surface-harmonics S_j' we have to substitute the above expression for V in the equation of equilibrium

$$\frac{\partial P}{\partial\xi}=\rho\left(\frac{\partial V}{\partial\xi}+\frac{\partial V'}{\partial\xi}+\frac{\partial\Omega}{\partial\xi}\right), \tag{15}$$

where Ω is used to denote the centrifugal potential.

§ 4. *The Centrifugal Potential.*—The equations of mechanical equilibrium (3) can be written in the form

$$\text{grad}\,P=\rho\,\text{grad}\left\{V+V'+\tfrac{1}{2}\omega^2(x^2+y^2)-\frac{M'}{M+M'}R\omega^2 x\right\}. \tag{16}$$

We see at once that the centrifugal potential can be written as

$$\Omega=\tfrac{1}{3}\omega^2 r^2(1-P_2(\mu))-\frac{M'}{M+M'}R\omega^2 r P_1(\sin\theta\cos\phi), \tag{17}$$

where $P_j(x)$ is used to denote the Legendre polynomial of order j in x. Using equation (2) defining ω^2 we can rewrite the above as

$$\Omega=\tfrac{1}{3}\omega^2\alpha^2\xi^2(1-P_2(\mu))-\frac{GM'}{R}\left(\frac{\alpha}{R}\right)\xi P_1(\sin\theta\cos\phi)+\text{constant}. \tag{18}$$

§ 5. *The Outer Potential* V'.—It is shown in II (§ 2) that consistent with our present scheme of approximation we can treat the secondary as a mass point and that, further, it is sufficient to retain the first four terms in the

expansion of r'^{-1} * in terms of the Legendre polynomials. We have (*cf.* II, equation (5))

$$V' = \frac{GM'}{R}\sum_{j=1}^{4}\left(\frac{\alpha}{R}\right)^j \xi^j P_j(\sin\theta\cos\phi) + \text{constant}. \tag{19}$$

By (18) and (19) we have

$$V' + \Omega = \frac{GM'}{R}\sum_{j=2}^{4}\left(\frac{\alpha}{R}\right)^j . \xi^j . P_j(\sin\theta\cos\phi) + \tfrac{1}{3}\omega^2\alpha^2\xi^2(1 - P_2(\mu)). \tag{20}$$

§ 6. Introducing (20) in (15) with the expression for V given by (14) we find that in the summation

$$\Sigma\xi^j S_j'(\mu, \phi)$$

only the surface harmonics

$$P_2(\cos\theta); \qquad P_j(\sin\theta\cos\phi), \qquad j = 2, 3, 4, \tag{21}$$

occur and with the numerical coefficients

$$\tfrac{1}{6}; \qquad -\tfrac{1}{2}\frac{M'}{M+M'}\left(\frac{\alpha}{R}\right)^{j-2}, \qquad j = 2, 3, 4, \tag{22}$$

respectively.

Hence our expression for V takes finally the form

$$V = (n+1) . K . \lambda^{\frac{1}{n}}\left[\Theta + v\left\{\tfrac{1}{6}\xi^2(P_2(\mu) - 1) - \sum_{j=2}^{4}\frac{1}{2}\frac{M'}{M+M'}\cdot\left(\frac{a}{R}\right)^{j-2} . \xi^j . P_j(\sin\theta\cos\phi)\right\}\right]. \tag{23}$$

To determine the precise forms of the surface harmonics $S_j(\mu, \phi)$ occurring in the expression for Θ, we have to make the above expression for V and also its derivative continuous with an expression of the type

$$V_{\text{outer}} = (n+1) . K . \lambda^{\frac{1}{n}}\left[\frac{C_0}{\xi} + v\sum_{j=1}^{\infty}\frac{S_j''(\mu, \phi)}{\xi^{j+1}}\right], \tag{24}$$

on a sphere of radius ξ_1, the first zero of θ. On going through the procedure we find that among the S_j's only the surface harmonics

$$P_2(\mu); \qquad P_j(\sin\theta\cos\phi), \qquad j = 2, 3, 4, \tag{25}$$

occur and with the numerical coefficients

$$-\tfrac{5}{6}\frac{\xi_1^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)} \tag{26}$$

and

$$\tfrac{1}{2}\frac{M'}{M+M'}\cdot\left(\frac{\alpha}{R}\right)^{j-2}\cdot\frac{(2j+1)\xi_1^j}{(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)}, \qquad j = 2, 3, 4, \tag{26'}$$

respectively.

We finally obtain therefore as the solution of the problem

$$\Theta = \theta + v\left\{\psi_0(\xi) - \tfrac{5}{6}\cdot\frac{\xi_1^2 P_2(\cos\theta)\psi_2(\xi)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)} + \sum_{j=2}^{4}(j+\tfrac{1}{2})\cdot\frac{M'}{M+M'}\cdot v^{j-2}\cdot\frac{\xi_1^2 P_j(\sin\theta\cos\phi)\psi_j(\xi)}{(j+1)\psi_j(\xi_1) + \xi_1\psi_j'(\xi_1)}\right\}, \tag{27}$$

* r' is the distance of the centre of gravity of the secondary to the point (x, y, z) in the primary.

where (as in II) we have introduced the quantity ν defined as the ratio of the radius of the undistorted configuration to the distance R between the components, *i.e.*,

$$\nu = \frac{a\xi_1}{R}. \tag{28}$$

We therefore see that the structure of the component will be specified completely to the order of accuracy with which we are concerned (i.e. *to the fifth order in* ν) when the functions ψ_0, ψ_2, ψ_3, and ψ_4 are known. ψ_0 and ψ_2 are tabulated in I for certain values of n, and ψ_3, ψ_4 are tabulated in II for the same values of n. Further quadratures are unnecessary.

§ 6. *Mass and Volume Relations.*—The mass is given by

$$M = \iiint \rho r^2 dr d\mu d\phi.$$

All the terms in the Legendre polynomials occurring in our solution for Θ do not contribute to the mass. The actual calculation proceeds exactly as in I, § 8, and yields

$$M = 4\pi a^3 \xi_1^2 \,.\, |\, \theta_1' \,|\,.\, \lambda \,.\, \left[1 + v \frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\, \theta_1' \,|} \right]$$

or a "mass-relation"

$$M_\omega = M_0 \,.\, \left[1 + v \,.\, \frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\, \theta_1' \,|} \right], \tag{29}$$

an expression identical in form with the corresponding one obtained for the simple rotational problem—the Kepler angular velocity ω here replacing the uniform angular velocity occurring in that problem. Similarly we have a volume-relation and a relation between the mean and the central density of the same form as in the rotational problem (*cf.* I, § 9):

$$N_\omega = N_0 \,.\, \left[1 + v \frac{3\psi_0(\xi_1)}{\xi_1 \,|\, \theta_1' \,|} \right], \tag{30}$$

$$\rho_{\text{mean}} = -\lambda \,.\, \frac{3}{\xi_1} \left(\frac{d\theta}{d\xi} \right)_1 \,.\, \left[1 + v \frac{\frac{1}{3}\xi_1^2 - \psi_0'(\xi_1)\xi_1 - 3\psi_0(\xi_1)}{\xi_1 \,.\, |\, \theta_1' \,|} \right]. \tag{31}$$

Again, by definition

$$v = \frac{\omega^2}{2\pi G\lambda}, \tag{32}$$

and since we are working only to the first order in v, we can rewrite the above as

$$v = \frac{(M+M')}{MR^3} \cdot \frac{M}{2\pi\lambda} = 2\left(1 + \frac{M'}{M} \right) \cdot \nu^3 \cdot \frac{|\, \theta_1' \,|}{\xi_1}. \tag{33}$$

§ 7. *Comparison with the Rotation and Tidal Problems.*—Using the above expression for v (equation (33)), we can rewrite our solution (27) for Θ as

$$\Theta = \theta + v \left\{ \psi_0(\xi) - \tfrac{5}{6} \frac{\xi_1^2 P_2(\cos\theta) \,.\, \psi_2(\xi)}{3\psi_2(\xi_1) + \xi_1 \psi_2'(\xi_1)} \right\}$$
$$+ \xi_1 \,.\, |\, \theta_1' \,|\,.\, \frac{M'}{M} \sum_{j=2}^{4} \Delta_j \,.\, \nu^{j+1} \,.\, P_j(\sin\theta \cos\phi) \,.\, \frac{\psi_j(\xi)}{\psi_j(\xi_1)}, \tag{34}$$

where (as in II) we have introduced the quantities Δ_j defined by

$$\Delta_j = \frac{(2j+1)\psi_j(\xi_1)}{(j+1)\psi_2(\xi_1)+\xi_1\psi_j'(\xi_1)}, \qquad j=2,\ 3,\ 4. \tag{35}$$

On comparing * the above solution with the corresponding solutions for the simple rotational and the tidal problems (I, equation (36) and II, equation (30)), we notice that the "distortion terms" (*i.e.* $\Theta-\theta$) occurring in (34) are exactly the sum of the corresponding terms in those two problems. In other words, *the distortion of a "double-star" component is the same as if it rotated like a rigid body about its own axis with the Kepler angular velocity* ω *and then tidally influenced by a secondary at a distance* R *from its centre of gravity, the two effects being simply added. This superposition theorem is true to the fifth order in the ratio of the radius of the component to the distance between the two components.*†

§ 8. *External Shape.*—The boundary ξ_0 is given by $\Theta=0$, or by (34)

$$\xi_0 = \xi_1 + \frac{v}{|\theta_1'|}\left\{\psi_0(\xi_1) - \tfrac{5}{6}\frac{\xi_1^2\psi_2(\xi_1)P_2(\mu)}{3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)}\right\} + \xi_1\frac{M'}{M}\sum_{j=2}^{4}\Delta_j \,.\, \nu^{j+1} \,.\, P_j(\sin\theta\cos\phi). \tag{36}$$

On comparing this equation of the boundary with those obtained for the simple rotational and tidal problems, we again meet here another example of the superposition theorem stated in § 7. (36) can be written more conveniently as (on using (33))

$$\frac{\xi_0-\xi_1}{\xi_1} = 2\left(1+\frac{M'}{M}\right) \,.\, \nu^3 \,.\, \frac{\psi_0(\xi_1)}{\xi_1^2} - \tfrac{1}{3}\left(1+\frac{M'}{M}\right) \,.\, \Delta_2 \,.\, \nu^3 \,.\, P_2(\cos\theta) + \frac{M'}{M}\sum_{j=2}^{4}\Delta_j \,.\, \nu^{j+1} \,.\, P_j(\sin\theta\cos\phi). \tag{37}$$

We thus see that there is an expansion of the configuration as a whole of amount

$$2\left(1+\frac{M'}{M}\right)\nu^3 \,.\, \frac{\psi_0(\xi_1)}{\xi_1^2}, \tag{38}$$

and superposed on this general expansion are the "ellipticity" and "harmonic" terms. The *deviations* from a spherical shape are given therefore only by the terms in $P_2(\cos\theta)$ and $P_j(\sin\theta\cos\phi)$. We shall therefore introduce the function $\sigma(\theta, \phi)$ ‡ defined as

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* In comparing the solution with that of the tidal problem we have to remember that the secondary in the tidal problem was placed on the Z-axis, while now the other component of our double-star system is on the X-axis. Naturally the "cos θ" in the tidal problem is now replaced by "sin θ cos ψ."

† This theorem is *not* likely to be true for higher orders in ν, for the sixth and higher order terms ought to take explicitly into account the density distribution in the secondary. I am indebted to Dr. B. Strömgren for this remark.

‡ $\sigma(\theta)$ has the simple meaning that it represents the fractional deviations from a *spherical* volume having the same volume as our distorted configuration.

$$\sigma(\theta, \phi) = \frac{\xi_0 - \xi_1}{\xi_1} - 2\left(1 + \frac{M'}{M}\right) . \nu^3 . \frac{\psi_0(\xi_1)}{\xi_1^2}$$

$$= -\tfrac{1}{3}\left(1 + \frac{M'}{M}\right) . \Delta_2 . \nu^3 . P_2(\cos\theta) + \frac{M'}{M}\sum_{j=2}^{4} \Delta_j . \nu^{j+1} . P_j(\sin\theta \cos\phi). \quad (39)$$

A general idea of the nature of the distortion in shape suffered by the configuration can be obtained from a study of the function $\sigma(\theta, \phi)$. (A reference

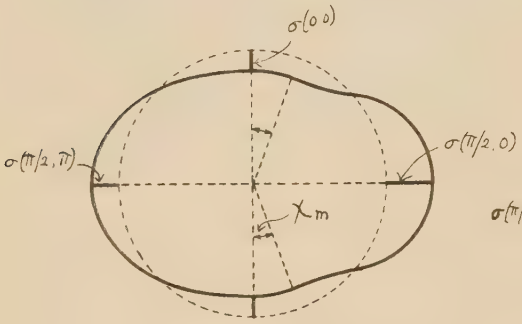

Fig. 1
The Meridian Plane, $\sigma(\theta, 0)$

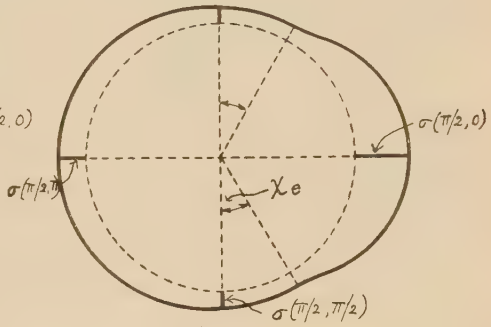

Fig. 2
The Equatorial Plane, $\sigma(\pi/2, \phi)$

to figs. 1 and 2 may help to visualise the nature of the surface we are considering.) In the *equatorial plane* $\sigma(\theta, \phi)$ has the form

$$\sigma\left(\frac{\pi}{2}, \phi\right) = \tfrac{1}{6}\left(1 + \frac{M'}{M}\right)\nu^3 + \frac{M'}{M}\sum_{j=2} \Delta_j . \nu^{j+1} . P_j(\cos\phi). \quad (40)$$

In the *principal meridian* through the centre of gravity of the secondary we have

$$\sigma(\theta, 0) = -\tfrac{1}{3}\left(1 + \frac{M'}{M}\right) . \Delta_2 . \nu^3 . P_2(\cos\theta) + \frac{M'}{M} . \sum_{j=2}^{4} \Delta_j . \nu^{j+1} . P_j(\sin\theta), \quad (41)$$

and finally in the *diametrical plane* along the axis of y we have

$$\sigma\left(\theta, \frac{\pi}{2}\right) = -\tfrac{1}{3}\left(1 + \frac{M'}{M}\right) . \Delta_2 . \nu^3 . P_2(\cos\theta) - \tfrac{1}{2} . \frac{M'}{M} . \Delta_2 . \nu^3 + \tfrac{3}{8} . \frac{M'}{M} . \Delta_4 . \nu^5. \quad (42)$$

From the above equations we easily deduce that

$$\left.\begin{aligned}
\sigma\left(\frac{\pi}{2}, 0\right) &= \tfrac{1}{6}\left(1 + 7\frac{M'}{M}\right) . \Delta_2 . \nu^3 + \frac{M'}{M} . \Delta_3 . \nu^4 + \frac{M'}{M} . \Delta_4 . \nu^5, \\
\sigma\left(\frac{\pi}{2}, \pi\right) &= \tfrac{1}{6}\left(1 + 7\frac{M'}{M}\right) . \Delta_2 . \nu^3 - \frac{M'}{M} . \Delta_3 . \nu^4 + \frac{M'}{M} . \Delta_4 . \nu^5, \\
\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) &= \tfrac{1}{6}\left(1 - 2\frac{M'}{M}\right) . \Delta_2 . \nu^3 + \tfrac{3}{8} . \frac{M'}{M} . \Delta_4 . \nu^5, \\
\sigma(0, 0) &= -\tfrac{1}{3}\left(1 + \tfrac{5}{2}\frac{M'}{M}\right) . \Delta_2 . \nu^3 + \tfrac{3}{8} . \frac{M'}{M} . \Delta_4 . \nu^5.
\end{aligned}\right\} \quad (43)^*$$

* We may notice the inequality $\sigma\left(\frac{\pi}{2}, 0\right) > \sigma\left(\frac{\pi}{2}, \pi\right) > \sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) > \sigma(0, 0)$. At $(\pi/2, 0)$ we have the *absolute maximum* and at $(0, 0)$ or $(\pi, 0)$ the *absolute minimum*.

The exact numerical forms of equations (39) to (43) for the various polytropes ($n = 1$, 1·5, 2, 3 and 4) can be read out at once on referring to Table VI in II where the values of Δ_2, Δ_3, Δ_4 for some values of n are given. To obtain the actual fractional elongations or contractions at the various points $\left(\theta = \frac{\pi}{2},\ \phi = 0, \text{ etc.}\right)$ we have to add to the various values of σ obtained from (43) the general expansion term (38). To obtain this (as also the precise numerical forms of the equations of the boundary for different values of n) we need the value of $\psi_0(\xi_1)/\xi_1^2$. Values of $\psi_0(\xi_1)$ are given in Table VI, I. For convenience the following short table is added:—

n	1	1·5	2	3	4
$\frac{\psi_0(\xi_1)}{\xi_1^2}$	0·10132	0·09694	0·10108	0·12273	0·14961

(44)

Again from (43) we see that *if we neglect quantities of the order* ν^4 *and* ν^5 *then the equilibrium configurations are ellipsoids. The ellipticities* ϵ_m, ϵ_e *and* ϵ_d *in the principal meridian, the equatorial plane and in the diametrical plane along the axis of* y *respectively are given by*

$$\left.\begin{aligned} \epsilon_m &= \sigma\left(\frac{\pi}{2}, 0\right) - \sigma(0, 0) &&= \tfrac{1}{2}\left(1 + 4\frac{M'}{M}\right)\Delta_2 \cdot \nu^3 \\ \epsilon_e &= \sigma\left(\frac{\pi}{2}, 0\right) - \sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) &&= \tfrac{3}{2}\frac{M'}{M}\Delta_2 \cdot \nu^3 \\ \epsilon_d &= \sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) - \sigma(0, 0) &&= \tfrac{1}{2}\left(1 + \frac{M'}{M}\right)\Delta_2 \cdot \nu^3 \end{aligned}\right\}. \qquad (43')$$

Also we notice that

$$\left.\begin{aligned} \epsilon_m > \epsilon_d > \epsilon_e \quad &\text{if} \quad 2M' < M \\ \epsilon_m > \epsilon_e \geqslant \epsilon_d \quad &\text{if} \quad 2M' \geqslant M \end{aligned}\right\}. \qquad (43'')$$

There is still another feature of the external shape that has to be discussed. Now if we take the equation of the boundary in the meridian plane specified by (41) (apart from the constant additive term (38)) then we find that the equation

$$\frac{d\sigma(\theta, 0)}{d\theta} = 0, \qquad (45)$$

has a solution apart from $\theta = 0$, $\frac{\pi}{2}$ or π. Equation (45) explicitly takes the form

$$\left(1 + \frac{M'}{M}\right) \cdot \Delta_2 \cdot \sin\theta + \frac{M'}{M}\{3\Delta_2 \sin\theta + \tfrac{1}{2}\Delta_3 \cdot \nu \cdot (15\sin^2\theta - 3) + \tfrac{1}{2}\Delta_4 \cdot \nu^2(35\sin^3\theta - 15\sin\theta)\} = 0. \qquad (45')$$

An approximate solution of (45′) when θ is small can be obtained by omitting in the above all terms in $\sin^2\theta \,.\, \nu$ or ν^3 and writing θ for $\sin\theta$. If χ_m denotes the solution, then

$$\chi_m = \tfrac{3}{2}\frac{\Delta_3}{\Delta_2}\frac{M'}{M+4M'} \,.\, \nu + O(\nu^3). \tag{46}$$

Similarly, if we take the equatorial plane we again have a solution for

$$\frac{d\sigma\left(\frac{\pi}{2}, \phi\right)}{d\phi} = 0, \tag{47}$$

apart from $\phi = 0, \pi/2, \pi, 3\pi/2$. The equation determining this angle is found to be

$$3\Delta_2 \cos\phi + \tfrac{1}{2} \,.\, \Delta_3 \,.\, \nu \,.\, (15\cos^2\phi - 3) + \tfrac{1}{2}\Delta_4 \,.\, \nu^2(35\cos^3\phi - 15\cos\phi) = 0. \tag{47′}$$

Denoting the solution by $(\pi/2 - \chi_e)$ we obtain as an approximate solution:

$$\chi_e = \tfrac{1}{2} \,.\, \frac{\Delta_3}{\Delta_2} \,.\, \nu + O(\nu^3). \tag{48}$$

This angle is the same as the angle at which the furrow is formed in the tidal problem (*cf.* II, § 8). In the following table the values of $\dfrac{\Delta_3}{\Delta_2}$ for different values of n are given so that χ_m and χ_e can be obtained from (46) and (48) in their numerical forms:—

n	1	1·5	2	3	4
Δ_3/Δ_2	0·79803	0·85937	0·91344	0·97904	0·99780

(49)

§ 9. Finally, one rather curious fact may be noted. In the double-star problem there can actually be a genuine fractional contraction in the equatorial plane, if the ratio of the mass of the secondary to that of the primary is greater than a certain quantity which we will now determine. Now, the fractional change at $(\theta = \pi/2, \phi = \pi/2)$ in the equatorial plane is given by (*cf.* equation (43))

$$\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) + 2\left(1 + \frac{M'}{M}\right) \,.\, \nu^3 \,.\, \frac{\psi_0(\xi_1)}{\xi_1^2} = \left\{2\left(1 + \frac{M'}{M}\right) \,.\, \frac{\psi_0(\xi_1)}{\xi_1^2} + \tfrac{1}{6}\left(1 - \frac{2M'}{M}\right) \,.\, \Delta_2 + O(\nu)\right\}\nu^3. \tag{50}$$

We will neglect terms of $O(\nu)$ compared to unity. Then the above expression is negative if

$$\frac{M'}{M} > \tfrac{1}{2}\frac{\Delta_2 + 12\dfrac{\psi_0(\xi_1)}{\xi_1^2}}{\Delta_2 - 6\dfrac{\psi_0(\xi_1)}{\xi_1^2}}. \tag{51}$$

It is easily verified that the quantity on the right is just the ratio of the fractional elongation at the equator to the fractional contraction at the poles that would occur in the simple rotational problem where the configuration is rotating about the Z-azis with a constant, uniform angular velocity. Or more precisely stated, *if the ratio of the mass of the secondary to that of the primary in our double-star system is equal to the ratio of the fractional elongation at the equator to the fractional contraction at the poles, which would be found if the primary rotated about itself as a rigid body with a uniform angular velocity, then the radius of the configuration in the equatorial plane and in a direction at right angles to the line joining the centres of gravity of the two components suffers no change in length.* If the ratio M'/M is greater than this, then there is contraction in this direction in the equatorial plane.

§ 10. *Change in the Central Density.*—In I, § 13, we considered the change in the central density as the configuration is set rotating. A similar question can be asked in connection with the double-star problem as well. Remembering that the mass-relation in the double-star problem is identical in form with that in the simple rotational problem, we find that the fractional change in the central density is given by (I, equation (61)),

$$\frac{\delta\lambda}{\lambda} = -v\frac{2n}{3-n}\frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\theta_1'|}. \tag{52}$$

Numerical forms of (52) have already been obtained in I. Again, by (33) we can rewrite (52) as

$$\frac{\delta\lambda}{\lambda} = -\left(1+\frac{M'}{M}\right).\frac{4n}{3-n}.\frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{\xi_1}.\nu^3. \tag{53}$$

Thus in the double-star problem we meet with the same difference in the behaviour of polytropes with $n > 3$ and those with $n < 3$, as was met with in the simple rotational problem. There is an increase in the central density if $n > 3$ and a decrease if $n < 3$.

In a separate paper the general properties of the three classes of distorted polytropes we have studied will be more closely examined to obtain information on the qualitative behaviour of the tidal and rotational distortions regarded as functions of the complete march of the density distribution in the configuration.

In conclusion, I have great pleasure in recording my thanks to Professor N. Bohr for allowing the very valuable privileges of his Institute, where the above work was carried out.

Institut for Teoretisk Fysik,
Copenhagen :
1933 *April* 2.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE NIZAMIAH OBSERVATORY, HYDERABAD, DURING THE YEAR 1932.

(Communicated by the Director)

No.	Star	Mag.	Phase	Date 1932	G.C.T. h m s
1	α Vir	1·2	D	Jan. 1	22 35 15·7
2	α Vir	1·2	R	1	23 45 40·3
3	16 Tau	5·4	D	18	17 50 38·9
4	q Tau	4·3	D	18	17 52 39·8
5	21 Tau	5·8	D	18	18 20 14·6
6	20 Tau	4·1	D	18	18 23 42·6
7	22 Tau	6·5	D	18	18 24 55·2
8	c Gem	5·5	D	22	21 4 54·0
9	43 B Lib	5·7	R	30	22 0 50·5
10	34 Leo	6·4	D	Apr. 16	18 30 40·6
11	π Sco	3·0	D	22	23 35 4·4
12	π Sco	3·0	R	23	0 47 32·9
13	γ Cap	3·7	D	Oct. 9	16 37 7·6
14	σ Aqr	4·8	D	10	18 48 24·9
15	η Cap	4·8	D	Dec. 2	14 24 17·0
16	31 B Vir	6·4	R	20	21 37 40·8

All the observations were made by Mr. M. K. Bappu (M) with the 15-inch Grubb refractor. The times were noted with a stop-watch and the clock corrections have been determined from observations with the Transit Instrument.

Observer's remarks :

13, 14, instantaneous.
8, sky partly cloudy.
16, seen through clouds.
7, time slightly uncertain.

The reductions have been carried out by Mr. M. V. V. Sastri according to the method adopted by the *Nautical Almanac* office, and the results are given below. As suggested by Professor Brown, a correction of $0^h{\cdot}0030$ has been added to the G.C.T., to allow for the error of 6″ in the mean longitude of the Moon.

No.	χ ° ′	ρ ° ′	$\sigma' - \sigma$ ″	Cos $(\chi - \rho)$
1	112 50	116 43	− 1·3	+0·998
2	330 48	116 36	+ 2·0	−0·827
3	136 52	74 32	+ 0·7	+0·464
4	92 3	74 32	− 2·2	+0·954
5	78 48	74 38	− 0·7	+0·997
6	123 55	74 38	− 0·1	+0·652
7	86 2	74 39	− 2·3	+0·980
8	129 7	102 2	− 0·1	+0·890

No.	χ	ρ	$\sigma' - \sigma$	Cos $(\chi - \rho)$
	° ′	° ′	″	
9	256 15	110 20	(−13·2)	−0·828
10	90 5	115 22	+ 0·6	+0·904
11	103 17	103 54	− 2·9	+1·000
12	272 26	103 32	(+11·8)	−0·981
13	46 42	66 26	− 2·1	+0·941
14	65 40	63 22	− 1·7	+0·999
15	125 21	69 6	+ 1·9	+0·556
16	322 33	118 15	+ 0·6	−0·911

Nizamiah Observatory,
Hyderabad, India.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE TOMSK UNIVERSITY OBSERVATORY DURING THE YEAR 1932.

N. Goryatscheff.

(Communicated by the Secretaries)

Date (G.C.T.)	Star	Mag.	Phase	Loc. Sid. Time	Δt
				h m s	s
Mar. 14·605782	B.D. +26° 750	7·8	D	7 40 05·2	+ 3·9
,, 15·612574	B.D. +28° 871	8·2	D	7 53 50·2	+10·4
,, 15·696578	B.D. +28° 902	8·4	D	9 55 08·1	+ 8·9
,, 15·738780	406 B Tauri	5·6	D	10 56 04·3	+ 7·0
,, 15·746005	B.D. +28° 918	8·0	D	11 06 30·3	+ 6·2
Apr. 9·676468	17 Tauri	3·8	D	11 04 39·7	+ 7·8
,, 9·695225	23 Tauri	4·3	D	11 31 44·7	+ 7·6
,, 11·638017	B.D. +27° 778	8·7	D	10 17 01·4	+ 4·4
,, 11·730786	B.D. +27° 799	8·1	D	12 30 58·7	+ 2·6
,, 13·721352	B.D. +26° 1554	8·5	D	12 25 14·5	+ 8·2
,, 13·752384	B.D. +26° 1564	7·3	D	13 10 03·0	+ 6·4
June 16·678602	*b* Scorpii	4·7	D	15 35 50·4	+ 5·6

Apparent Places of Occulted Stars

	h m s	° ′ ″	
B.D. +26° 750	4 44 08·12	+26 51 00·9	Gr_{1910} 2040
B.D. +28 871	5 40 42·39	+28 10 10·4	Gr_{1910} 2564
B.D. +28 902	5 44 24·37	+28 18 19·7	Gr_{1910} 2614
B.D. +28 918	5 47 16·32	+28 04 41·7	Gr_{1910} 2652
B.D. +27 778	5 25 14·98	+27 47 35·4	Gr_{1910} 2376
B.D. +27 799	5 30 21·77	+27 51 14·9	Gr_{1910} 2423
B.D. +26 1554	7 22 09·03	+26 05 36·1	Gr_{1910} 3771
B.D. +26 1564	7 23 51·34	+26 22 06·8	Gr_{1910} 3787

Instruments: Zeiss telescope, 90-mm. aperture; and chronometer of Ericcson.

Observations with eye and ear.

Observer: Prof. N. Goryatscheff.

The proper motions of stars are computed from Gr_{1910} Catalogue.

Tomsk University Observatory:
1933 *March* 7.

ERRATUM.

Annual Report, pp. 305–317, Heading of President's Address, *for* on the Award of the Gold Medal, *read* on the Distances and Motions of the Extra-Galactic Nebulæ.